Pseudo-hoops

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Abstract

In this paper we study the pseudo-hoops, structures introduced by B. Bosbach in [6, 7] under the name of complementary semigroups. We prove some of their properties and we define the basic concepts of filter and normal filter. The lattice of normal filters is isomorphic with the lattice of congruences of a pseudo-hoop. We also study some important classes of pseudo-hoops. Bounded Wajsberg pseudo-hoops are equivalent to pseudo-Wajsberg algebras and bounded basic pseudo-hoops are equivalent to pseudo-BL algebras. Some examples of pseudo-hoops are given in the last section of the paper.

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Introduction

Hoops are naturally ordered commutative residuated integral monoids, introduced by B. Bosbach in [6, 7], then studied by J.R. Büchi and T.M. Owens in [8], a paper never published. All information about this paper is taken from

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[1, 3, 4]. In the last years, hoops theory was enriched with deep structure theorems (see [4, 15, 2, 3, 5, 1]). Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops ([1], Corollary 2.10) one obtains an elegant short proof of the completeness theorem for propositional basic logic (see [1], Theorem 3.8), introduced by Hàjek in [21]. The algebraic structures corresponding to Hàjek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops. The main example of BL-algebra is the interval $[0, 1]$ endowed with the structure induced by a t-norm. MV-algebras, product algebras and Gödel algebras are the most known classes of BL-algebras. Recent investigations are concerned with non-commutative generalizations for these structures. Pseudo-MV algebras were introduced as a non-commutative generalization of MV-algebras (see [18, 19]). Equivalent structures were defined and studied in [9, 10], under the name of pseudo-Wajsberg algebras. Pseudo-Wajsberg algebras are a non-commutative version of Wajsberg algebras. A. Dvurecenskij proved in [14] that the category of pseudo-MV algebras is equivalent to the category of l-groups with strong unit. This theorem extends the fundamental result established by D. Mundici for the commutative case [22].

In [12, 20], pseudo-BL algebras were defined as a common extension of BL-algebras and pseudo-MV algebras. The main source of examples of pseudo-BL algebras is l-group theory. In [16], there was introduced a notion of pseudo-t-norm in order to recapture some of the properties of pseudo-BL algebras. For the interval $[0, 1]$, this notion induces some more general algebras named weak pseudo-BL algebras.

The aim of this paper is to study the pseudo-hoops, structures introduced by B. Bosbach in [6, 7] under the name of complementary semigroups. Pseudo BL-algebras will appear as particular cases of pseudo-hoops.

The paper is divided into four sections. In the first section we recall some facts concerning residuated structures. In Section 2 we study the pseudo-hoops and we prove their basic properties. Following ideas from [18, 19, 12, 13], in Section 3 we define filters and normal filters, and we prove that the lattice of normal filters and the lattice of congruences of a pseudo-hoop are isomorphic. In Section 4 we investigate some classes of pseudo-hoops, namely, cancellative pseudo-hoops, Wajsberg pseudo-hoops, basic pseudo-hoops, product pseudo-hoops, and (strongly) simple pseudo-hoops. The most important of these classes are Wajsberg pseudo-hoops and basic pseudo-hoops. We show that bounded Wajsberg pseudo-hoops are equivalent to pseudo-Wajsberg algebras and that bounded basic pseudo-hoops are equivalent to pseudo-BL algebras. These facts generalize results from [4] and [1]. In the last section of the paper we give some examples of pseudo-hoops and normal filters.

1 Preliminaries

Firstly, we shall recall some facts about residuated and complemented monoids. For details see [6, 7, 4, 23]. A structure $(A, \odot, 1, \leq)$ is a *partially ordered monoid (po-monoid)* if
(i) $(A, \odot, 1)$ is a monoid;
(ii) $\leq$ is a partial order on A;
(iii) for all $a, b, c \in A$,
$a \leq b$ implies $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$.
Assume $(A, \odot, 1, \leq)$ is a po-monoid. Then $(A, \odot, 1, \leq)$ is *integral* if $a \leq 1$ for all $a \in A$.
The largest element (under $\leq$) of the set $\{c \in A \mid c \odot a \leq b\}$, if it exists, is called the *left-residual of* a *relative to* b, and is denoted by $a \to b$. Thus $a \to b$ can be defined by the condition
$\forall x (x \odot a \leq b \Leftrightarrow x \leq a \to b)$.
$(A, \odot, 1, \leq)$ is *left-residuated* if $a \to b$ exists for all $a, b \in A$. In this event the enriched structure $(A, \odot, \to, 1, \leq)$ is called a *left-residuated po-monoid.* A left-residuated po-monoid can be thought of as an algebra $(A, \odot, \to, 1)$, since the partial order can be retrieved via $a \leq b$ iff $a \to b = 1$.
The *inverse right divisibility relation* $\leq_r$ on a monoid $(A, \odot, 1)$ is defined by
$a \leq_r b \Leftrightarrow \exists c (a = c \odot b)$.
An algebra $(A, \odot, \to, 1)$ is a *left-complemented monoid* if $(A, \odot, \to, 1, \leq_r)$ is a po-monoid with left-residuation $\to$.
The notions of *right-residual*, *right-residuated po-monoid*, *inverse left divisibility relation* $\leq_l$, and *right-complemented monoid* are defined similarly.

Lemma 1.1 ([4], Lemma 1.3)
(i) If $(A, \odot, \to, 1)$ is a left-complemented monoid, then $\leq_r$ is a meet-semilattice order, where $a \wedge b = (a \to b) \odot a$ for all $a, b \in A$.
(i$'$) If $(A, \odot, \to, 1)$ is a right-complemented monoid, then $\leq_l$ is a meet-semilattice order, where $a \wedge b = a \odot (a \to b)$ for all $a, b \in A$.

Left- (and right-) complemented monoids form a variety and have a simple equational characterization.

Proposition 1.2 ([7]; [4], Theorem 1.4)
An algebra $\mathbf{A} = (A, \odot, \to, 1,)$ is a left-complemented monoid iff the following identities hold:
(i) $a \odot 1 = 1 \odot a = a$;
(ii) $a \to a = 1$;
(iii) $(a \to b) \odot a = (b \to a) \odot b$;
(iv) $(a \odot b) \to c = a \to (b \to c)$.

Dually, we get

Proposition 1.3 An algebra $\mathbf{A} = (A, \odot, \rightarrow, 1,)$ is a right-complemented monoid iff the following identities hold:
(i) $a \odot 1 = 1 \odot a = a$;
(ii) $a \rightarrow a = 1$;
(iii) $a \odot (a \rightarrow b) = b \odot (b \rightarrow a)$;
(iv) $(a \odot b) \rightarrow c = b \rightarrow (a \rightarrow c)$.

If the underlying monoid is commutative, the notions of left- and right- complemented monoid coincide. Commutative (left-)complemented monoids are called *hoops* by Büchi and Owens in [8].

2 Basic definitions and properties

A *pseudo-hoop* is an algebra $\mathbf{A} = (A, \odot, \rightarrow, \rightsquigarrow, 1)$ with three binary operations $\odot, \rightarrow, \rightsquigarrow$ and one constant 1 such that:
(i) $(A, \odot, \rightarrow, 1)$ is a left-complemented monoid;
(i) $(A, \odot, \rightsquigarrow, 1)$ is a right-complemented monoid;
(iii) $\leq_r = \leq_l$.

In the sequel, we shall agree that $\odot$ has priority towards the operations $\rightarrow, \rightsquigarrow$. Sometimes, for the sake of clarity, we shall put parenthses even if this is not necessary.
By (iii), the inverse left and right divisibility relations coincide in a pseudo-hoop and will be denoted simply by $\leq$. It follows that for any $a, b \in A$,
$a \leq b$ iff $\exists c(a = b \odot c)$ iff $\exists c(a = c \odot b)$.
A *linear* (or *totally ordered*) pseudo-hoop is a pseudo-hoop with the property that $\leq$ is a total order.
A *bounded* pseudo-hoop is an algebra $\mathbf{A} = (A, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ such that $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-hoop and $0 \leq a$ for all $a \in A$.
We shall denote the set of natural numbers by ω. We define $a^0 = 1$, $a \overset{0}{\rightarrow} b = a \overset{0}{\rightsquigarrow} b = b$ and
$a^n = a^{n-1} \odot a, \qquad a \overset{n}{\rightarrow} b = a \rightarrow (a \overset{n-1}{\rightarrow} b), \qquad a \overset{n}{\rightsquigarrow} b = a \rightsquigarrow (a \overset{n-1}{\rightsquigarrow} b)$
for $n \in \omega - \{0\}$.
In the following propositions, we collect and reformulate some results proved by B. Bosbach in [6, 7]. For the sake of completeness, we shall include proofs of these results.

Proposition 2.1 [6, 7] Let $\mathbf{A} = (A, \odot, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-hoop.
(i) $a \leq b$ iff $a \rightarrow b = 1$ iff $a \rightsquigarrow b = 1$.
(ii) $(A, \leq)$ is a meet-semilattice, with $a \wedge b = (a \rightarrow b) \odot a = a \odot (a \rightsquigarrow b)$.
(iii) $a \rightarrow a = a \rightsquigarrow a = 1$;
(iv) $(a \rightarrow b) \odot a = (b \rightarrow a) \odot b = a \odot (a \rightsquigarrow b) = b \odot (b \rightsquigarrow a)$;
(v) $a \odot b \rightarrow c = a \rightarrow (b \rightarrow c)$ and $a \odot b \rightsquigarrow c = b \rightsquigarrow (a \rightsquigarrow c)$.

Proof: (i) Since $\leq = \leq_r = \leq_l$, we have that $(A, \odot, \to, 1, \leq)$ is a left-residuated monoid, hence $a \leq b$ iff $a \to b = 1$, and $(A, \odot, \rightsquigarrow, 1, \leq)$ is a right-residuated monoid, so $a \leq b$ iff $a \rightsquigarrow b = 1$.
(ii) By Lemma 1.1.
(iii) By Propositions 1.2(ii) and 1.3(ii).
(iv) By (ii) and the fact that $a \wedge b = b \wedge a$.
(iv) By Propositions 1.2(iv) and 1.3(iv). □

Theorem 2.2 [6, 7] An algebra $\mathbf{A} = (A, \odot, \to, \rightsquigarrow, 1)$ of type $(2, 2, 2, 0)$ is a pseudo-hoop iff the following identities hold:
(A0) $a \odot 1 = 1 \odot a = a$;
(A1) $a \to a = a \rightsquigarrow a = 1$;
(A2) $a \odot b \to c = a \to (b \to c)$;
(A3) $a \odot b \rightsquigarrow c = b \rightsquigarrow (a \rightsquigarrow c)$ and ;
(A4) $(a \to b) \odot a = (b \to a) \odot b = a \odot (a \rightsquigarrow b) = b \odot (b \rightsquigarrow a)$.

Proof: By the above proposition, the identities (A0)-(A4) hold in any pseudo-hoop. Conversely, suppose that $\mathbf{A} = (A, \odot, \to, \rightsquigarrow, 1)$ satisfies (A0)-(A4). By Propositions 1.2 and 1.3, we get that $(A, \odot, \to, 1)$ is a left-complemented monoid and $(A, \odot, \rightsquigarrow, 1)$ is a right-complemented monoid. Use Lemma 1.1 and (A4) to get that $\leq_3 = \leq_l$. Hence, $\mathbf{A}$ is a pseudo-hoop. □

From the above theorem it follows that pseudo-hoops form a variety. We shall denote by $\mathcal{PH}$ this variety.

Remark 2.3 [6, 7] Let $\mathbf{A} = (A, \odot, \to, \rightsquigarrow, 1)$ be a pseudo-hoop. Then $\odot$ is commutative iff $\to = \rightsquigarrow$. In this case, $\mathbf{A}$ is a hoop.

Proof: If $\odot$ is commutative, then for any $a, b, c \in A$ we have that $c \leq a \to b$ iff $c \odot a \leq b$ iff $a \odot c \leq b$ iff $c \leq a \rightsquigarrow b$. Hence, $a \to b = a \rightsquigarrow b$. Conversely, suppose that $\to = \rightsquigarrow$ and let $a, b \in A$. Then for any $c \in A$, $a \odot b \leq c$ iff $a \leq b \to c$ iff $a \leq b \rightsquigarrow c$ iff $b \odot a \leq c$. That is, $a \odot b = b \odot a$.
We get that $(A, \odot, \to, 1)$ is a commutative left-complemented monoid, that is a hoop. □

In the sequel we shall prove some properties of pseudo-hoops.

Lemma 2.4 [6, 7] Let $\mathbf{A} = (A, \odot, \to, \rightsquigarrow, 1)$ be a pseudo-hoop. For any $a, b, c \in A$, the following hold:
(1) $c \odot a \leq b$ iff $c \leq a \to b$;
(2) $a \odot c \leq b$ iff $c \leq a \rightsquigarrow b$;
(3) $1 \to a = 1 \rightsquigarrow a = a$;
(4) $a \to 1 = a \rightsquigarrow 1 = 1$;
(5) $a \to b \leq (c \to a) \to (c \to b)$;
(6) $a \rightsquigarrow b \leq (c \rightsquigarrow a) \rightsquigarrow (c \rightsquigarrow b)$;
(7) for any $n \in \omega$, $a \xrightarrow{n} b = a^n \to b$ and $a \overset{n}{\rightsquigarrow} b = a^n \rightsquigarrow b$.

Proof: (1)-(2) By the fact that $\rightarrow$ is left-residuation and $\rightsquigarrow$ is right-residuation.
(3)-(6) See [4], Lemma 1.5 and its dual.
(7) see [4], pag. 554. $\square$

Lemma 2.5 [6, 7] Let $\mathbf{A} = (A, \odot, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-hoop. For any $a, b, c \in A$,
(8) $a \odot b \leq a, b$;
(9) $a \leq b \rightarrow a$ and $a \leq b \rightsquigarrow a$;
(10) $a \leq b$ implies $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$;
(11) $a \odot b \leq a \wedge b$;
(12) $a \leq b$ implies $c \rightarrow a \leq c \rightarrow b$ and $c \rightsquigarrow a \leq c \rightsquigarrow b$;
(13) $a \leq b$ implies $b \rightarrow c \leq a \rightarrow c$ and $b \rightsquigarrow c \leq a \rightsquigarrow c$;
(14) $(b \rightarrow c) \odot (a \rightarrow b) \leq a \rightarrow c$;
(15) $(a \rightsquigarrow b) \odot (b \rightsquigarrow c) \leq a \rightsquigarrow c$;
(16) $a \rightarrow b \leq (b \rightarrow c) \rightsquigarrow (a \rightarrow c)$;
(17) $a \rightsquigarrow b \leq (b \rightsquigarrow c) \rightarrow (a \rightsquigarrow c)$;
(18) $a \rightarrow b \leq (a \odot c) \rightarrow (b \odot c)$;
(19) $a \rightsquigarrow b \leq (c \odot a) \rightsquigarrow (c \odot b)$.

Proof: (8) By (A3), (A1) and (4), $a \odot b \rightsquigarrow a = b \rightsquigarrow (a \rightsquigarrow a) = b \rightsquigarrow 1 = 1$. Similarly, applying (A2), (A1) and (4), $a \odot b \rightarrow b = a \rightarrow (b \rightarrow b) = a \rightarrow 1 = 1$. Apply now Proposition 2.1(i) to get that $a \odot b \leq a$ and $a \odot b \leq b$.
(9) Apply (8) and (1), respectively (2).
(10) Since $a \leq b$, there are $x, y \in A$ such that $a = x \odot b$ and $a = b \odot y$. It follows that $a \odot c = x \odot (b \odot c)$ and $c \odot a = (c \odot b) \odot y$, hence $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$.
(11) We have that $a \wedge b = (a \rightarrow b) \odot a$, by Proposition 2.1(ii), and $b \leq a \rightarrow b$, by (9). Applying (10), we get that $a \odot b \leq a \odot (a \rightarrow b) = a \wedge b$.
(12) Apply (5), (6), and the fact that $a \leq b$ iff $a \rightarrow b = a \rightsquigarrow b = 1$.
(13) Suppose that $a \leq b$. By (10), it follows that $(b \rightarrow c) \odot a \leq (b \rightarrow c) \odot b = b \wedge c \leq c$. Apply now (1) to get that $b \rightarrow c \leq a \rightarrow c$. Similarly, $a \odot (b \rightsquigarrow c) \leq b \odot (b \rightsquigarrow c) = b \wedge c \leq c$, so $b \rightsquigarrow c \leq a \rightsquigarrow c$, by (2).
(14) Apply (5) to get that $b \rightarrow c \leq (a \rightarrow b) \rightarrow (a \rightarrow c)$, and (1) to obtain $(b \rightarrow c) \odot (a \rightarrow b) \leq a \rightarrow c$.
(15) Similarly, applying (6) and (2).
(16) By (14) and (2).
(17) By (15) and (1).
(18) Applying (1) and the fact that $a \wedge b = (a \rightarrow b) \odot a$, we get that $a \rightarrow b \leq (a \odot c) \rightarrow (b \odot c)$ iff $(a \rightarrow b) \odot a \odot c \leq b \odot c$ iff $(a \wedge b) \odot c \leq b \odot c$, which is true by (10).
(19) Applying (2) and the fact that $a \wedge b = a \odot (a \rightsquigarrow b)$, we have that $a \rightsquigarrow b \leq (c \odot a) \rightsquigarrow (c \odot b)$ iff $c \odot a \odot (a \rightsquigarrow b) \leq c \odot b$ iff $c \odot (a \wedge b) \leq c \odot b$, which is true by (10). $\square$

Lemma 2.6 Let $\mathbf{A}$ be a pseudo-hoop and I an arbitrary set. Then

(20) $b \to \bigwedge_{i\in I} a_i = \bigwedge_{i\in I}(b \to a_i)$;
(21) $b \rightsquigarrow \bigwedge_{i\in I} a_i = \bigwedge_{i\in I}(y \to a_i)$,
whenever the arbitrary meets exist.

Proof: (20) For any $x \in A$ we have the following equivalences:
$x \leq b \to \bigwedge_{i\in I} a_i$ iff $x \odot b \leq \bigwedge_{i\in I} a_i$ iff $x \odot b \leq a_i$, for any $i \in I$ iff $x \leq b \to a_i$, for any $i \in I$ iff $x \leq \bigwedge_{i\in I}(b \to a_i)$.
(21) Similarly. $\Box$

Lemma 2.7 Let $\mathbf{A}$ be a pseudo-hoop. For any $a, b, c, d \in A$,
(22) $(a \to b) \odot (c \to d) \leq (a \wedge c) \to (b \wedge d)$;
(23) $(a \rightsquigarrow b) \odot (c \rightsquigarrow d) \leq (a \wedge c) \rightsquigarrow (b \wedge d)$;

Proof: (22) By (13), we have that $a \to b \leq (a\wedge c) \to b$ and $c \to d \leq (a\wedge c) \to d$. Hence, by (10) and (11), $(a \to b) \odot (c \to d) \leq ((a \wedge c) \to b) \odot ((a \wedge c) \to d) \leq ((a\wedge c) \to b)\wedge((a\wedge c) \to d)$. Applying (20), we get that $((a\wedge c) \to b)\wedge((a\wedge c) \to d) = (a \wedge c) \to (b \wedge d)$.
(23) Similarly. $\Box$

The following result extends Lemma 2.1 from [1].

Proposition 2.8 Let $\mathbf{A}$ be a pseudo-hoop and $a, b, c \in A$. Suppose that $a \vee b$ exists. Then
(i) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$;
(i$'$) $(a \vee b) \rightsquigarrow c = (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$;
(ii) for any $n \in \omega - \{0\}$,
$(a \vee b)^n \to c = \bigwedge\{(x_n \odot \ldots \odot x_1) \to c \mid x_i \in \{a, b\}\}$;
(ii$'$) for any $n \in \omega - \{0\}$,
$(a \vee b)^n \rightsquigarrow c = \bigwedge\{(x_1 \odot \ldots \odot x_n) \rightsquigarrow c \mid x_i \in \{a, b\}\}$.

Proof: (i) Since $a, b \leq a \vee b$, by (13) we get that $(a \vee b) \to c \leq a \to c$ and $(a\vee b) \to c \leq b \to c$, so $(a\vee b) \to c \leq (a \to c)\wedge(b \to c)$. If $x \leq (a \to c)\wedge(b \to c)$, then $x \leq a \to c$ and $x \leq b \to c$, so $x \odot a \leq c$ and $x \odot b \leq c$, hence $a, b \leq x \rightsquigarrow c$. But this implies $a \vee b \leq x \rightsquigarrow c$, that is $x \odot (a \vee b) \leq c$, so $x \leq (a \vee b) \to c$. We have got that $x \leq (a \to c) \wedge (b \to c)$ implies $x \leq (a \vee b) \to c$. It follows that $(a \to c) \wedge (b \to c) \leq (a \vee b) \to c$.
(i$'$) Similarly.
(ii) For $n = 1$, we get (i). Assume that the equality holds for n. Then
$(a \vee b)^{n+1} \to c = (a \vee b) \odot (a \vee b)^n \to c \overset{(A2)}{=} (a \vee b) \to ((a \vee b)^n \to c) = (a \vee b) \to \bigwedge\{(x_n \odot \ldots \odot x_1) \to c \mid x_i \in \{a, b\}\} \overset{(i)}{=} (a \to \bigwedge\{(x_n \odot \ldots \odot x_1) \to c \mid x_i \in \{a, b\}\}) \wedge (b \to \bigwedge\{(x_n \odot \ldots \odot x_1) \to c \mid x_i \in \{a, b\}\}) \overset{(20)}{=} \bigwedge\{a \to ((x_n \odot \ldots \odot x_1) \to c) \mid x_i \in \{a, b\}\} \wedge \bigwedge\{b \to ((x_n \odot \ldots \odot x_1) \to c) \mid x_i \in \{a, b\}\} \overset{(A2)}{=} \bigwedge\{(a \odot x_n \odot \ldots \odot x_1) \to c \mid x_i \in \{a, b\}\} \wedge \bigwedge\{(b \odot x_n \odot \ldots \odot x_1) \to c \mid x_i \in \{a, b\}\} = \bigwedge\{(x_{n+1} \odot x_n \odot \ldots \odot x_1) \to c \mid x_i \in \{a, b\}\}$.
(ii$'$) Similarly. $\Box$

Lemma 2.9 Let **A** be a pseudo-hoop and I an arbitrary set. Then
(i) $a \odot (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \odot b_i)$;
(i$'$) $(\bigvee_{i\in I} b_i) \odot a = \bigvee_{i\in I}(b_i \odot a)$;
(ii) $a \wedge (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \wedge b_i)$,
whenever the arbitrary unions exist.

Proof: (i) We have that $b_i \leq \bigvee_{i\in I} b_i$ for any $i \in I$, so $a \odot b_i \leq a \odot (\bigvee_{i\in I} b_i)$ for any $i \in I$, hence $\bigvee_{i\in I}(a \odot b_i) \leq a \odot (\bigvee_{i\in I} b_i)$. For any $x \in A$, we get that $\bigvee_{i\in I}(a \odot b_i) \leq x$ implies $a \odot b_i \leq x$ for any $i \in I$, so $b_i \leq a \rightsquigarrow x$ for any $i \in I$. Thus, we obtain that $\bigvee_{i\in I} b_i \leq a \rightsquigarrow x$, hence $a \odot (\bigvee_{i\in I} b_i) \leq x$. We have got that $\bigvee_{i\in I}(a \odot b_i) \leq x$ implies $a \odot (\bigvee_{i\in I} b_i) \leq x$. It follows that $a \odot (\bigvee_{i\in I} b_i) \leq \bigvee_{i\in I}(a \odot b_i)$.
(i$'$) Similarly.
(ii) The inequality $\bigvee_{i\in I}(a \wedge b_i) \leq a \wedge (\bigvee_{i\in I} b_i)$ is obvious. Let us now prove the converse inequality. We have that $a \wedge (\bigvee_{i\in I} b_i) = (\bigvee_{i\in I} b_i) \wedge a = (\bigvee_{i\in I} b_i) \odot (\bigvee_{i\in I} b_i \rightsquigarrow a) = \bigvee_{i\in I}(b_i \odot (\bigvee_{i\in I} b_i \rightsquigarrow a))$, by (i$'$). Applying (13), we get that $\bigvee_{i\in I} b_i \rightsquigarrow a \leq b_i \rightsquigarrow a$ for any $i \in I$, so $b_i \odot (\bigvee_{i\in I} b_i \rightsquigarrow a) \leq b_i \odot (b_i \rightsquigarrow a) = b_i \wedge a = a \wedge b_i$ for any $i \in I$. It follows that $\bigvee_{i\in I}(b_i \odot (\bigvee_{i\in I} b_i \rightsquigarrow a)) \leq \bigvee_{i\in I}(a \wedge b_i)$.
$\square$

Lemma 2.10 Let **A** be a pseudo-hoop. For any $a, b \in A$,
(24) $a \leq (a \to b) \rightsquigarrow b$;
(25) $a \leq (a \to b) \rightsquigarrow a$;
(26) $a \leq (a \rightsquigarrow b) \to b$;
(27) $a \leq (a \rightsquigarrow b) \to a$;
(28) $b \to ((a \to b) \rightsquigarrow a) = b \to a$;
(29) $b \rightsquigarrow ((a \rightsquigarrow b) \to a) = b \rightsquigarrow a$;
(30) $((b \to a) \rightsquigarrow a) \to a = b \to a$;
(31) $((b \rightsquigarrow a) \to a) \rightsquigarrow a = b \rightsquigarrow a$;
(32) $[((b \to a) \rightsquigarrow a) \to b] \to (b \to a) = b \to a$;
(33) $[((b \rightsquigarrow a) \to a) \rightsquigarrow b] \rightsquigarrow (b \rightsquigarrow a) = b \rightsquigarrow a$.

Proof: (24)-(25) From $(a \to b) \odot a = a \wedge b \leq a, b$ and (2) we obtain that $a \leq (a \to b) \rightsquigarrow b$ and $a \leq (a \to b) \rightsquigarrow a$.
(26)-(27) Similarly, $a \odot (a \rightsquigarrow b) = a \wedge b \leq a, b$, so $a \leq (a \rightsquigarrow b) \to b$ and $a \leq (a \rightsquigarrow b) \to a$, by (1).
(28) Let us denote $(a \to b) \rightsquigarrow a$ by x. We have to prove that $b \to x = b \to a$. By (25), we have that $a \leq x$. Hence, applying (12) we get that $b \to a \leq b \to x$. From $a \leq x$ and (13) we obtain that $x \to b \leq a \to b$ and, by (24), $a \to b \leq ((a \to b) \rightsquigarrow a) \to a = x \to a$. Thus, we have got that $x \to b \leq x \to a$. It follows that $(b \to x) \odot b = b \wedge x = x \wedge b = (x \to b) \odot x \leq (x \to a) \odot x = x \wedge a \leq a$, hence $b \to x \leq b \to a$, by (1).
(29) Similarly.
(30) By (26), $b \to a \leq ((b \to a) \rightsquigarrow a) \to a$. From (24) we get that $b \leq (b \to$

$a) \rightsquigarrow a$, and applying (13), it follows that $((b \to a) \rightsquigarrow a) \to a \le b \to a$.
(31) Similarly.
(32) Let $x = (b \to a) \rightsquigarrow a$. By (24) and (30), we have that $b \le x$ and $x \to a = b \to a$. It follows that $(x \to b) \to (b \to a) = (x \to b) \to (x \to a) = (x \to b) \odot x \to a = (x \wedge b) \to a = b \to a$.
(33) Similarly. $\Box$

Remember that on any hoop $\mathbf{A}$ one can define a *pseudo-join* operation $\dot{\vee}$ (see [1]) by

$a \dot{\vee} b = ((a \to b) \to b) \wedge ((b \to a) \to a)$, for all $a, b \in A$.

Following this idea, we define on a pseudo-hoop $\mathbf{A}$ two binary operations that are almost a join operation. If $a, b \in A$, then the *pseudo-joins* of a and b are

$a \vee_1 b = ((a \to b) \rightsquigarrow b) \wedge ((b \to a) \rightsquigarrow a)$,
$a \vee_2 b = ((a \rightsquigarrow b) \to b) \wedge ((b \rightsquigarrow a) \to a)$.

Proposition 2.11 Let $\mathbf{A}$ be a pseudo-hoop. For any $a, b \in A$,
(i) $a \vee_1 b = b \vee_1 a$ and $a \vee_2 b = b \vee_2 a$;
(ii) $a, b \le a \vee_1 b$ and $a, b \le a \vee_2 b$;
(iii) $a \le b$ iff $a \vee_1 b = b$;
(iii$'$) $a \le b$ iff $a \vee_2 b = b$.

Proof: (i) is obvious.
(ii) By (24), we have that $a \le (a \to b) \rightsquigarrow b$ and by (9), $a \le (b \to a) \rightsquigarrow a$. Hence, $a \le a \vee_1 b$. Similarly, applying (26) and (9), we get that $a \le a \vee_2 b$.
(iii) If $a \le b$, then $(a \to b) \rightsquigarrow b = 1 \rightsquigarrow b = b$, by (3). Hence, $a \vee_1 b = b \wedge [(b \to a) \rightsquigarrow a] = b$, since $b \le (b \to a) \rightsquigarrow a$, by (24). Conversely, suppose that $a \vee_1 b = b$. It follows that $a \wedge b = a \wedge (a \vee_1 b) = a$, by (ii). That is, $a \le b$.
(iii$'$) Similarly. $\Box$

Proposition 2.12 Let $\mathbf{A}$ be a pseudo-hoop. The following are equivalent:
(i) $\vee_1$ is associative;
(ii) for all $a, b, c \in A$, $a \le b$ implies $a \vee_1 c \le b \vee_1 c$;
(iii) for all $a, b, c \in A$, $a \vee_1 (b \wedge c) \le (a \vee_1 b) \wedge (a \vee_1 c)$;
(iv) $\vee_1$ is the join operation on A.

Proof: Similar to the proof of Proposition 2.4 from [1]. $\Box$

Dually, we get

Proposition 2.13 Let $\mathbf{A}$ be a pseudo-hoop. The following are equivalent:
(i) $\vee_2$ is associative;
(ii) for all $a, b, c \in A$, $a \le b$ implies $a \vee_2 c \le b \vee_2 c$;
(iii) for all $a, b, c \in A$, $a \vee_2 (b \wedge c) \le (a \vee_2 b) \wedge (a \vee_2 c)$;
(iv) $\vee_2$ is the join operation on A.

Remark 2.14 Suppose that $\vee_1$(respectively $\vee_2$) is associative. By Proposition 2.12, we get that $\vee_1$(respectively $\vee_2$) is the join operation on A. Applying Lemma 2.9(iii), it follows that $(A, \wedge, \vee)$ is a distributive lattice.

Proposition 2.15 Let $\mathbf{A}$ be a pseudo-hoop and $a, b, c \in A$. Then
(i) if $a \vee b = 1$, then $a \odot b = a \wedge b$;
(ii) if $a \vee b = 1$ and $a \leq c, b \leq d$, then $c \vee d = 1$;
(iii) if $a \vee b = 1$, then $a^n \vee b^n = 1$ for all $n \in \omega - \{0\}$.

Proof: (i) By (9), $a \odot b \leq a \wedge b$. Since, by Lemma 2.11(ii) we have that $a, b \leq a \vee_1 b$, it follows that $a \vee_1 b = 1$, that is $((a \to b) \rightsquigarrow b) \wedge ((b \to a) \rightsquigarrow a) = 1$, hence $(a \to b) \rightsquigarrow b = (b \to a) \rightsquigarrow a = 1$. It follows that $b \to a \leq a$, so $a \wedge b = (b \to a) \odot b \leq a \odot b$.
(ii) Of course, $c, d \leq 1$. Let $x \in A$ such that $c, d \leq x$. It follows that $a, b \leq x$, so $1 = a \vee b \leq x$. That is, $x = 1$.
(iii) We follow the proof of Lemma 2.16 from [12]. Suppose that $a \vee b = 1$. By Lemma 2.9(i), it follows that $a = a \odot 1 = a \odot (a \vee b) = a^2 \vee (a \odot b)$, so $1 = a \vee b = a^2 \vee (a \odot b) \vee b = a^2 \vee b$. Similarly, we get that $b = b \odot 1 = b \odot (a^2 \vee b) = (b \odot a^2) \vee b^2$. Hence, $a^2 \vee b^2 = a^2 \vee (b \odot a^2) \vee b^2 = a^2 \vee b = 1$. We prove in the same manner that $a^{2^n} \vee b^{2^n} = 1$ for all $n \in \omega - \{0\}$. Since $a^{2^n} \leq a^n, b^{2^n} \leq b^n$, it follows that $a^n \vee b^n = 1$. $\Box$

3 Filters and congruences

In this section we study the filters and the congruences of a pseudo-hoop. Following some ideas from [18, 19, 12, 13], we shall define the notion of normal filter and we shall establish an isomorphism between the lattice of normal filters and the lattice of congruences of a pseudo-hoop. The results obtained in this section generalize the similar results for pseudo-MV algebras [18, 19] and pseudo-BL algebras [12, 13].

Let $\mathbf{A} = (A, \odot, \to, \rightsquigarrow, 1)$ a pseudo-hoop. A non-empty subset F of A is a *filter* of $\mathbf{A}$ if for all $a, b \in A$,
(i) $a, b \in F$ implies $a \odot b \in F$;
(ii) $a \in F$ and $a \leq b$ imply $b \in F$.
By (11), it is obvious that any filter of $\mathbf{A}$ is also a filter of the meet-semilattice $(A, \wedge)$.
A filter F of $\mathbf{A}$ is *proper* iff $F \neq A$. A *maximal filter* (or *ultrafilter*) is a proper filter U of $\mathbf{A}$ that is not included in any other proper filter.

Proposition 3.1 For a subset F of A the following are equivalent:
(i) F is a filter;
(ii) $1 \in F$ and if $a, a \to b \in F$, then $b \in F$;
(iii) $1 \in F$ and if $a, a \rightsquigarrow b \in F$, then $b \in F$.

Proof: See [4], Theorem 1.6 and its dual. $\Box$

It follows that any filter of $\mathbf{A}$ is a also a subuniverse of $\mathbf{A}$.
If $X \subseteq A$, we denote by $< X >$ the filter generated by X in $\mathbf{A}$. A description of $< X >$ is easily obtained:

Proposition 3.2 Let $\mathbf{A}$ be a pseudo-hoop and $X \subseteq A$. Then
$< X >= \{a \in A \mid x_1 \odot x_2 \odot \ldots \odot x_n \leq b$ for some $n \in \omega - \{0\}$ and $x_1, \ldots, x_n \in X\}$

$= \{a \in A \mid x_1 \to (x_2 \to (\ldots \to (x_n \to a)\ldots)) = 1$ for some $n \in \omega - \{0\}$ and $x_1, \ldots, x_n \in X\}$

$= \{a \in A \mid x_1 \rightsquigarrow (x_2 \rightsquigarrow (\ldots \rightsquigarrow (x_n \rightsquigarrow a)\ldots)) = 1$ for some $n \in \omega - \{0\}$ and $x_1, \ldots, x_n \in X\}$

In particular the principal filter generated by an element $x \in A$ is

$< x >= \{a \in A \mid x^n \leq a$ for some $n \in \omega - \{0\}\} = \{a \in A \mid x \overset{n}{\to} a = 1$ for some $n \in \omega - \{0\}\} = \{a \in A \mid x \overset{n}{\rightsquigarrow} a = 1$ for some $n \in \omega - \{0\}\}$

Remark 3.3 Let $\mathbf{A}$ be a pseudo-hoop and F be a proper filter of $\mathbf{A}$. The following conditions are equivalent:
(i) F is a maximal filter;
(ii) for all $x \in A$, if $x \notin F$ then $< F \cup \{x\} >= A$.

Proof: (i)$\Rightarrow$(ii) We have that $F \subseteq < F \cup \{x\} >$ and $F \neq < F \cup \{x\} >$, since $x \notin F$. From the fact that F is maximal it follows that $< F \cup \{x\} >= A$.
(ii)$\Rightarrow$(i) Suppose that there exists a proper filter G of $\mathbf{A}$ such that $F \subseteq G$ and $F \neq G$. Then there is $x \in G \setminus F$. By (ii), we have that $< F \cup \{x\} >= A$. Since $< \cup\{x\} > \subseteq G$, it follows that $G = A$, hence G is not proper. $\Box$

Proposition 3.4 Let $\mathbf{A}$ be a pseudo-hoop and $a, b \in A$. If $a \vee b$ exists, then

$< a \vee b >=< a > \cap < b >$.

Proof: It is obvious that $a \in < a >$ and $b \in < b >$. Since $a, b \leq a \vee b$, it follows that $a \vee b \in < a >$ and $a \vee b \in < b >$, so $a \vee b \in < a > \cap < b >$. Hence, $< a \vee b > \subseteq < a > \cap < b >$. Conversely, let $c \in < a > \cap < b >$. Then $a^n \leq c$ and $b^m \leq c$ for some $n, m \in \omega - \{0\}$. By Proposition 2.8(ii$'$), we get that

$(a \vee b)^{n+m} \rightsquigarrow c = \bigwedge\{(x_1 \odot \ldots \odot x_{n+m}) \rightsquigarrow c \mid x_i \in \{a, b\}\}$.

Consider $x_1, \ldots, x_{n+m} \in \{a, b\}$. Denote by r the number of occurences of a in the sequence $x_1, \ldots, x_{n+m}$ and by s the number of occurences of b in the sequence $x_1, \ldots, x_{n+m}$. Of course, $r + s = n + m$. We have that $x_1 \odot \ldots \odot x_{n+m} \leq a^r$ and $x_1 \odot \ldots \odot x_{n+m} \leq b^s$. Applying (13), it follows that $a^r \rightsquigarrow c \leq (x_1 \odot \ldots \odot x_{n+m}) \rightsquigarrow c$ and $b^s \rightsquigarrow c \leq (x_1 \odot \ldots \odot x_{n+m}) \rightsquigarrow c$. If $r \leq n$, then $s \geq m$, so $b^s \leq b^m$. Applying again (13), we get that $b^s \rightsquigarrow c \geq b^m \rightsquigarrow c = 1$, since $b^m \leq c$. Hence, $b^s \rightsquigarrow c = 1$, so $x_1 \odot \ldots \odot x_{n+m} \rightsquigarrow c = 1$. Similarly, if $r > n$, then $a^r \leq a^n$, so $a^r \rightsquigarrow c \geq a^n \rightsquigarrow c = 1$, since $a^n \leq c$. It follows that $a^r \rightsquigarrow c = 1$, hence $(x_1 \odot \ldots \odot x_{n+m}) \rightsquigarrow c = 1$. Therefore, $(x_1 \odot \ldots \odot x_{n+m}) \rightsquigarrow c = 1$ for any $x_1, \ldots, x_{n+m} \in \{a, b\}$, hence $(a \vee b)^{n+m} \rightsquigarrow c = 1$. Thus $(a \vee b)^{n+m} \leq c$, so $c \in < a \vee b >$. $\Box$

In a pseudo-BL algebra, one can introduce two distance functions in order to study the filters (see [12]). If $\mathbf{A}$ is a pseudo-hoop, then we define four *distance* functions:

$$d_1(a,b) = (a \to b) \odot (b \to a), \qquad d_2(a,b) = (a \rightsquigarrow b) \odot (b \rightsquigarrow a),$$
$$d_3(a,b) = (b \to a) \odot (a \to b), \qquad d_4(a,b) = (b \rightsquigarrow a) \odot (a \rightsquigarrow b).$$

Lemma 3.5 For any $a, b \in A$ and any $i \in \{1, 2, 3, 4\}$,
(i) $d_i(a,b) = 1$ iff $a = b$;
(ii) $d_i(a,a) = 1$;
(iii) $d_i(a,1) = a$;
(iv) if $i \in \{1,3\}$, then $d_i(b,c) \odot d_i(a,b) \odot d_i(b,c) \leq d_i(a,c)$;
(v) if $i \in \{2,4\}$, then $d_i(a,b) \odot d_i(b,c) \odot d_i(a,b) \leq d_i(a,c)$.

Proof: We shall prove the properties for $i = 1$, the other cases following similarly.
(i) We have that $d_1(a,b) = 1$ iff $a \to b = b \to a = 1$ iff $a \leq b$ and $b \leq a$ iff $a = b$.
(ii) By (A1), $d_1(a,a) = (a \to a) \odot (a \to a) = 1 \odot 1 = 1$.
(iii) By (3), (4) and (A0), $d_1(a,1) = (a \to 1) \odot (1 \to a) = 1 \odot a = a$.
(iv) We have that $d_1(b,c) \odot d_1(a,b) \odot d_1(b,c) = (b \to c) \odot (c \to b) \odot (a \to b) \odot (b \to a) \odot (b \to c) \odot (c \to b) = x \odot y$, where $x = (b \to c) \odot (c \to b) \odot (a \to b)$ and $y = (b \to a) \odot (b \to c) \odot (c \to b)$. By (8) and (14), we get that $x \leq (b \to c) \odot (a \to b) \leq a \to c$ and $y \leq (b \to a) \odot (c \to b) \leq c \to a$. Hence, applying (10) we get that $x \odot y \leq d_1(a,c)$.
(v) Similarly, applying (8) and (15). $\Box$

Let F be a filter of $\mathbf{A}$. We define two binary relations on A by:
$a \equiv_{R(F)} b \overset{def}{\Leftrightarrow} d_1(a,b) \in F$ and $a \equiv_{L(F)} b \overset{def}{\Leftrightarrow} d_2(a,b) \in F$.
We remark that for any $a, b \in A$,
$a \equiv_{R(F)} b$ iff $a \to b, b \to a \in F$ iff $d_3(a,b) \in F$, and
$a \equiv_{L(F)} b$ iff $a \rightsquigarrow b, b \rightsquigarrow a \in F$ iff $d_4(a,b) \in F$.

Proposition 3.6 Let $\mathbf{A}$ be a pseudo-hoop. For a given filter F of $\mathbf{A}$, the relations $\equiv_{R(F)}$ and $\equiv_{L(F)}$ are equivalence relations on A.

Proof: Let us prove that $\equiv_{R(F)}$ is an equivalence relation on A. By Lemma 3.5(ii), $d_1(a,a) = 1 \in F$, hence $a \equiv_{R(F)} a$ for any $a \in A$. It is obvious that $\equiv_{R(F)}$ is symmetric. It remains to prove the transitivity of $\equiv_{R(F)}$. Suppose that $a \equiv_{R(F)} b$ and $b \equiv_{R(F)} c$ for some $a, b, c \in A$. Hence, $d_1(a,b), d_1(b,c) \in F$. By Lemma 3.5(iv), $d_1(b,c) \odot d_1(a,b) \odot d_1(b,c) \leq d_1(a,c)$, hence $d_1(a,c) \in F$. That is, $a \equiv_{R(F)} c$. We prove similarly that $\equiv_{L(F)}$ is an equivalence relation on A. $\Box$

Lemma 3.7 $F = \{a \in A \mid a \equiv_{R(F)} 1\} = \{a \in A \mid a \equiv_{L(F)} 1\}$.

Proof: We have that $a \equiv_{R(F)} 1$ iff $d_1(a,1) \in F$ iff $a \in F$, since $d_1(a,1) = a$. Similarly, $a \equiv_{L(F)} 1$ iff $d_2(a,1) \in F$ iff $a \in F$, since $d_2(a,1) = a$. $\Box$

Proposition 3.8 Let F be a filter of $\mathbf{A}$. Then for all $a, b \in A$,
(i) $a \equiv_{R(F)} b$ iff $x \odot a = y \odot b$ for some $x, y \in F$;
(i$'$) $a \equiv_{L(F)} b$ iff $a \odot x = b \odot y$ for some $x, y \in F$.

Proof: (i) Suppose that $x \odot a = y \odot b$, for some $x, y \in F$. By (A2), we get that $y \to (b \to a) = (y \odot b) \to a = (x \odot a) \to x = 1$, since $x \odot a \leq x$. It follows that $y \leq b \to a$, hence $b \to a \in F$, since $y \in F$. Similarly, we get that $x \to (a \to b) = 1$, hence $a \to b \in F$. We have got that $a \to b, b \to a \in F$, that is $a \equiv_{R(F)} b$. Conversely, suppose that $a \equiv_{R(F)} b$, hence $a \to b, b \to a \in F$. Let $x = a \to b$ and $y = b \to a$. We have that $x, y \in F$ and $x \odot a = (a \to b) \odot a = a \wedge b = b \wedge a = (b \to a) \odot b = y \odot b$.
(i$'$) Similarly. □

Let us denote by $A/R(F)$ ($A/L(F)$, respectively) the quotient set associated with $\equiv_{R(F)}$ ($\equiv_{L(F)}$, respectively). For any $a \in A$, $a/R(F)$ ($a/L(F)$, respectively) will denote the equivalence class of a with respect to $\equiv_{R(F)}$ ($\equiv_{L(F)}$, respectively).

Lemma 3.9 Let F be a filter of $\mathbf{A}$. Then for all $a, b, c, d \in A$,
(i) if $a \equiv_{R(F)} b$ and $c \equiv_{R(F)} d$, then $a \to c \in F$ iff $b \to d \in F$;
(i$'$) if $a \equiv_{L(F)} b$ and $c \equiv_{L(F)} d$, then $a \rightsquigarrow c \in F$ iff $b \rightsquigarrow d \in F$.

Proof: (i) We have that $a \to b, b \to a, c \to d, d \to c \in F$. By (14), we get that $(c \to d) \odot (a \to c) \odot (b \to a) \leq (a \to d) \odot (b \to a) \leq b \to d$ and $(d \to c) \odot (b \to d) \odot (a \to b) \leq (b \to c) \odot (a \to b) \leq a \to c$. From the fact that F is filter, it follows that $a \to c \in F$ iff $b \to d \in F$.
(i$'$) Similarly. □

The previous lemma allows us to define the binary relation $\leq_{R(F)}$ on $A/R(F)$ by:

$a/R(F) \leq_{R(F)} b/R(F) \overset{def}{\Leftrightarrow} a \to b \in F$.

It is straightforward to prove that $\leq_{R(F)}$ is an order relation on $A/R(F)$. Similarly, we define an order relation $\leq_{L(F)}$ on $A/L(F)$ by:

$a/L(F) \leq_{L(F)} b/L(F) \overset{def}{\Leftrightarrow} a \rightsquigarrow b \in F$.

In the sequel, we shall introduce normal filters in order to characterize the congruences of a pseudo-hoop $\mathbf{A}$.
A filter H of $\mathbf{A}$ is called *normal* if for every $a, b \in A$ we have the equivalence:

(N) $\quad a \to b \in H$ iff $a \rightsquigarrow b \in H$.

We remark that $\{1\}$ and A are normal filters of the pseudo-hoop $\mathbf{A}$.
For a filter F of $\mathbf{A}$ and $a \in A$ let us denote

$a \odot F = \{a \odot x \mid x \in F\}, \quad F \odot a = \{x \odot a \mid x \in F\}$.

Proposition 3.10 Let H be a a filter of $\mathbf{A}$. The following are equivalent:
(i) H is a normal filter;
(ii) $a \odot H = H \odot a$ for any $a \in A$;
(iii) $\equiv_{R(H)} = \equiv_{L(H)}$.

Proof: (i)$\Rightarrow$(ii). Let $a \in A$ and $y = a \odot x \in a \odot H$. It follows that $y = a \wedge y = (a \to y) \odot a$, since $y = a \odot x \leq a$. By (A3) and (A1), we have that

$x \rightsquigarrow (a \rightsquigarrow y) = (a \odot x) \rightsquigarrow (a \odot x) = 1$, so $x \leq a \rightsquigarrow y$, hence $a \rightsquigarrow y \in H$, since $x \in H$. From (N) we get that $a \to y \in H$. Thua, $y = (a \to y) \odot a \in H \odot a$. Hence, $a \odot H \subseteq H \odot a$. We prove similarly that $H \odot a \subseteq a \odot H$.
(ii)$\Rightarrow$(iii) Let $a, b \in A$ such that $a \equiv_{R(H)} b$. By Proposition 3.8(i), there are $x, y \in H$ such that $x \odot a = y \odot b$. Applying now (ii), there are $z, t \in H$ such that $x \odot a = a \odot z$ and $y \odot b = b \odot t$. Hence, $a \odot z = b \odot t$ for some $z, t \in H$. Applying Proposition 3.8(i$'$), we obtain that $a \equiv_{L(H)} b$. Similarly, $a \equiv_{L(H)} b$ implies $a \equiv_{R(H)} b$.
(iii)$\Rightarrow$(i) Let $a, b \in A$. By (20) and (A1), we get that $a \to a \wedge b = (a \to a) \wedge (a \to b) = 1 \wedge (a \to b) = a \to b$, hence $d_1(a, a \wedge b) = (a \to a \wedge b) \odot (a \wedge b \to a) = (a \to a \wedge b) \odot 1 = a \to a \wedge b = a \to b$. Similarly, using (21) we obtain that $d_2(a, a \wedge b) = a \rightsquigarrow b$. Applying (ii), it follows that $a \to b \in H$ iff $d_1(a, a \wedge b) \in H$ iff $a \equiv_{R(H)} a \wedge b$ iff $a \equiv_{L(H)} a \wedge b$ iff $d_2(a, a \wedge b) \in H$ iff $a \rightsquigarrow b \in H$. $\Box$

Lemma 3.11 Let H be a normal filter of $\mathbf{A}$ and $x \in A$. Then
$< H \cup \{x\} >= \{a \in A \mid h \odot x^n \leq a, \text{ for some } n \in \omega, h \in H\} = \{a \in A \mid x^n \odot h \leq a, \text{ for some } n \in \omega, h \in H\}$.

Proof: By Proposition 3.2, we have that
$< H \cup \{x\} >= \{a \in A \mid (h_1 \odot x^{n_1}) \odot (h_2 \odot x^{n_2}) \odot \ldots \odot (h_k \odot x^{n_k}) \leq a$ for some $k \in \omega - \{0\}, h_1, \ldots, h_n \in H, n_1, \ldots, n_k \in \omega\}$.
If $k = 1$, then we get $h_1 \odot x^{n_1} \leq a$.
If $k = 2$, then $a \geq (h_1 \odot x^{n_1}) \odot (h_2 \odot x^{n_2}) = h_1 \odot (x^{n_1} \odot h_2) \odot x^{n_2}$. Since H is normal, it follows that $x^{n_1} \odot h_2 \in x^{n_1} \odot H = H \odot x^{n_1}$, so there is $h_3 \in H$ such that $x^{n_1} \odot h_2 = h_3 \odot x^{n_1}$. We get that $a \geq h_1 \odot (x^{n_1} \odot h_2) \odot x^{n_2} = (h_1 \odot h_3) \odot (x^{n_1} \odot x^{n_2}) = h \odot x^n$, where $h = h_1 \odot h_3 \in H$ and $n = n_1 + n_2$. Applying repeatedly this procedure we obtain the intended result. $\Box$

Proposition 3.12 Let $\mathbf{A}$ be a pseudo-hoop and H be a proper normal filter of $\mathbf{A}$. The following conditions are equivalent:
(i) H is a maximal filter;
(ii) for all $x \in A$, if $x \notin H$ then for any $a \in A$, $x^n \to a \in F$ for some $n \in \omega$;
(ii$'$) for all $x \in A$, if $x \notin H$ then for any $a \in A$, $x^n \rightsquigarrow a \in F$ for some $n \in \omega$.

Proof: (i)$\Rightarrow$(ii) Let $x \in A$ such that $x \notin H$. By Remark 3.3, we have that $< H \cup \{x\} >= A$. Applying Lemma 3.11, we get that for all $a \in A$ there is $n \in \omega$ and $h \in H$ such that $h \odot x^n \leq a$, so $h \leq x^n \to a$. Since H is a filter of $\mathbf{A}$, it follows that $x^n \to a \in H$.
(ii)$\Rightarrow$(i) Let $x \in A$ such that $x \notin H$. For any $a \in A$, there is $n \in \omega$ such that $x^n \to a \in H$. We get that $(x^n \to a) \odot x^n = x^n \wedge a \leq a$, so $a \in < H \cup \{x\} >$, by Lemma 3.11. Hence, $< H \cup \{x\} >= A$. Apply now Remark 3.3 to obtain that H is maximal.
(ii$'$)$\Leftrightarrow$(i) Similarly. $\Box$

By Proposition 3.10, if H is a normal filter of $\mathbf{A}$, then $\equiv_{R(H)}$ and $\equiv_{L(H)}$ coincide.

We shall denote by $\equiv_H$ this equivalence relation and by a/H the equivalence class of $a \in A$.

Proposition 3.13 Let H be a a normal filter of $\mathbf{A}$. Then $\equiv_H$ is a congruence on $\mathbf{A}$.

Proof: Suppose that $a \equiv_H b$ and $c \equiv_H d$. By Proposition 3.8(i), there are $x, y \in A$ such that $x \odot a = y \odot b$. It follows that $x \odot (a \odot c) = y \odot (b \odot c)$, hence, applying again Proposition 3.8(i), $a \odot c \equiv_H b \odot c$. Similarly, from $c \equiv_H d$ and Proposition 3.8(i$'$) we get that $b \odot c \equiv_H b \odot d$. Thus, by the transitivity of $\equiv_H$, $a \odot c \equiv_H b \odot d$.
Let us prove now that $a \to c \equiv_H b \to d$. Firstly, we remark that $a \to b, b \to a, c \to d, d \to c \in H$. By (14), we have that $(c \to d) \odot (a \to c) \leq a \to d$ and $(d \to c) \odot (a \to d) \leq a \to c$. Applying now (1), we get that $c \to d \leq (a \to c) \to (a \to d)$ and $d \to c \leq (a \to d) \to (a \to c)$. Since H is a filter of $\mathbf{A}$, it follows that $(a \to c) \to (a \to d) \in H$ and $(a \to d) \to (a \to c) \in H$. Hence, $a \to c \equiv_H a \to d$. We prove in the same manner that $a \to d \equiv_H b \to d$. Thus, $a \to c \equiv_H b \to d$.
The compatibility of $\equiv_H$ with $\rightsquigarrow$ is proved in a similar way. $\Box$

Proposition 3.14 Let $\equiv$ be a congruence on $\mathbf{A}$ and $H_\equiv = \{a \in A \mid a \equiv 1\}$. Then $H_\equiv$ is a normal filter of $\mathbf{A}$.

Proof: We have that $1 \equiv 1$, hence $1 \in H_\equiv$. Let $a \in H_\equiv$ and $b \in A$ such that $a \leq b$, that is $a \to b = 1$. Since $a \equiv 1$ and $\equiv$ is a congruence on $\mathbf{A}$, we get that $1 = a \to b \equiv 1 \to b = b$, hence $b \in H_\equiv$. If $a, b \in H_\equiv$, then $a \equiv 1$ and $b \equiv 1$, so $a \odot b \equiv 1 \odot 1 = 1$, that is $a \odot b \in H_\equiv$. Thus, we have proved that $H_\equiv$ is a filter. Let us prove now that $H_\equiv$ satisfies condition (N).
Suppose that $a \to b \in H_\equiv$, that is $a \to b \equiv 1$. It follows that $a \wedge b = (a \to b) \odot a \equiv 1 \odot a = a$, so $a \rightsquigarrow b \equiv (a \wedge b) \rightsquigarrow b = 1$, hence $a \rightsquigarrow b \in H_\equiv$. We have got that $a \to b \in H_\equiv$ implies $a \rightsquigarrow b \in H_\equiv$. We prove similarly that $a \rightsquigarrow b \in H_\equiv$ implies $a \to b \in H_\equiv$. $\Box$

Proposition 3.15 The map $H \mapsto \equiv_H$ is an isomorphism between the lattice of normal filters of $\mathbf{A}$ and the lattice of congruences of $\mathbf{A}$. Its inverse is the map $\equiv \;\mapsto H_\equiv$.

Proof: Firstly, we shall prove that for any normal filter H of $\mathbf{A}$ and any congruence $\equiv$ of $\mathbf{A}$,

$\equiv_{H_\equiv} = \equiv$ and $H_{\equiv_H} = H$.

Let $a, b \in A$ such that $a \equiv b$. It follows that $a \to b \equiv b \to b = 1$ and $b \to a \equiv b \to b = 1$, hence $a \to b, b \to a \in H_\equiv$, that is $a \equiv_{H_\equiv} b$. Conversely, if $a \equiv_{H_\equiv} b$, then $a \to b \equiv b \to a \equiv 1$. We get that $a \wedge b = (a \to b) \odot a \equiv 1 \odot a = a$ and $a \wedge b = b \wedge a = (b \to a) \odot b \equiv 1 \odot b = b$, hence $a \equiv b$.
Finally, for all $a \in A$, we have that $a \in H_{\equiv_H}$ iff $a \equiv_H 1$ iff $a \to 1, 1 \to a \in H$ iff

$a, 1 \in H$ iff $a \in H$.
Thus, we have proved that $H \mapsto \equiv_H$ is a bijection between the normal filters of $\mathbf{A}$ and the congruences of $\mathbf{A}$. It is obvious that $H_1 \subseteq H_2$ iff $\equiv_{H_1} \subseteq \equiv_{H_2}$ for any normal filters H_1 and H_2. $\Box$

Proposition 3.16 The variety of pseudo-hoops is arithmetical.

Proof: Let us consider the following ternary terms: $p(x, y, z) = [(x \to y) \rightsquigarrow z] \wedge [(z \to y) \rightsquigarrow x]$ and $M(x, y, z) = [(y \to x) \rightsquigarrow x] \wedge [(z \to y) \rightsquigarrow y] \wedge [(x \to z) \rightsquigarrow z]$. Let $\mathbf{A}$ be a pseudo-hoop and $a, b \in A$. It follows that $p(a, a, b) = [(a \to a) \rightsquigarrow b] \wedge [(b \to a) \rightsquigarrow a] = (1 \rightsquigarrow b) \wedge [(b \to a) \rightsquigarrow a] = b \wedge [(b \to a) \rightsquigarrow a] = b$ and $p(a, b, b) = [(a \to b) \rightsquigarrow b] \wedge [(b \to b) \rightsquigarrow a] = [(a \to b) \rightsquigarrow b] \wedge a = a$, since $b \leq (b \to a) \rightsquigarrow a$ and $a \leq (a \to b) \rightsquigarrow b$, by (24). That is, p is a Mal'cev term for the variety of pseudo-hoops. We also have that $M(a, a, b) = [(a \to a) \rightsquigarrow a] \wedge [(b \to a) \rightsquigarrow a] \wedge [(a \to b) \rightsquigarrow b] = a \wedge [(b \to a) \rightsquigarrow a] \wedge [(a \to b) \rightsquigarrow b] = a$, since $a \leq (b \to a) \rightsquigarrow a$, by (9), and $a \leq (a \to b) \rightsquigarrow b$, by (24). We prove similarly that $M(a, b, a) = M(b, a, a) = a$. Thus, M is a majority term. $\Box$

4 Some classes of pseudo-hoops

Cancellative pseudo-hoops

A pseudo-hoop $\mathbf{A} = (A, \odot, \to, \rightsquigarrow, 1)$ is called *cancellative* if the monoid $(A, \odot, 1)$ is cancellative.

Proposition 4.1 A pseudo-hoop $\mathbf{A}$ is cancellative iff the following identities hold:
(C1) $b \to a \odot b = a$;
(C2) $b \rightsquigarrow b \odot a = a$.

Proof: Suppose that $\mathbf{A}$ is cancellative. It follows that $a \odot b = b \wedge (a \odot b) = (b \to a \odot b) \odot b$, hence $a = b \to a \odot b$. Similarly, $b \odot a = b \wedge (b \odot a) = b \odot (b \rightsquigarrow b \odot a)$, so $a = b \rightsquigarrow b \odot a$.
Conversely, suppose that $\mathbf{A}$ satisfies (C1) and (C2). Let $a, b, c \in A$. If $a \odot c = b \odot c$ then, applying twice (C1) we get that $a = c \to a \odot c = c \to b \odot c = b$. Similarly, from $c \odot a = c \odot b$ and (C2) it follows that $a = c \rightsquigarrow c \odot a = c \rightsquigarrow c \odot b = b$. $\Box$

By the above proposition, it follows that cancellative pseudo-hoops form a variety.

Proposition 4.2 Let $\mathbf{A}$ be a cancellative pseudo-hoop. For all $a, b, c \in A$,
(i) $c \to a = c \odot b \to a \odot b$;
(i$'$) $c \rightsquigarrow a = b \odot c \rightsquigarrow b \odot a$;
(ii) $a \odot b \leq c \odot b$ iff $a \leq c$;
(ii$'$) $b \odot a \leq b \odot c$ iff $a \leq c$.

Proof: (i) By (C1) and (A2), we have that $c \to a = c \to (b \to a \odot b) = c \odot b \to a \odot b$.
(i$'$) By (C2) and (A3), we get that $c \rightsquigarrow a = c \rightsquigarrow (b \rightsquigarrow b \odot a) = b \odot c \rightsquigarrow b \odot a$.
(ii) Applying (i), $a \odot b \leq c \odot b$ iff $a \odot b \to c \odot b = 1$ iff $a \to c = 1$ iff $a \leq c$.
(ii$'$) Similarly, applying (i$'$). $\Box$

Wajsberg pseudo-hoops

A pseudo-hoop $\mathbf{A} = (A, \odot, \to, \rightsquigarrow, 1)$ is called *Wajsberg* if it satisfies the following conditions
(W1) $(a \to b) \rightsquigarrow b = (b \to a) \rightsquigarrow a$;
(W2) $(a \rightsquigarrow b) \to b = (b \rightsquigarrow a) \to a$.

Proposition 4.3 Let $\mathbf{A}$ be a Wajsberg pseudo-hoop. Then
(i) $a \vee_1 b = (a \to b) \rightsquigarrow b = (b \to a) \rightsquigarrow a$ for all $a, b \in A$;
(i$'$) $a \vee_2 b = (a \rightsquigarrow b) \to b = (b \rightsquigarrow a) \to a$ for all $a, b \in A$;
(ii) $\vee_1$ and $\vee_2$ are associative;
(iii) $a \vee b = a \vee_1 b = a \vee_2 b$ for all $a, b \in A$.

Proof: (i) By the definition of $\vee_1$ and (W1).
(i$'$) By the definition of $\vee_2$ and (W2).
(ii) If $a \leq b$ and $c \in A$, then applying twice (13) we get that $b \to c \leq a \to c$ and $(a \to c) \rightsquigarrow c \leq (b \to c) \rightsquigarrow c$, that is $a \vee_1 c \leq b \vee_1 c$. By Proposition 2.12, $\vee_1$ is associative. We prove similarly that $\vee_2$ is associative.
(iii) Apply Remark 2.14. $\Box$

Bounded Wajsberg hoops are termwise definitionally equivalent to Wajsberg algebras (see [4], Theorem 1.19). A similar result is obtained in the case of pseudo-hoops.
A *pseudo-Wajsberg algebra* ([9]) is an algebra $\mathbf{A} = (A, \to, \rightsquigarrow, ^-, ^\sim, 1)$ with two binary operations $\to, \rightsquigarrow$, two unary operations $^-, ^\sim$ and one constant 1 satisfying the following axioms:
(i) $a \to a = a \rightsquigarrow a = 1$;
(ii) $(a \to b) \rightsquigarrow b = (b \to a) \rightsquigarrow a = (b \rightsquigarrow a) \to a = (a \rightsquigarrow b) \to b$;
(iii) $(a \to b) \to [(b \to c) \rightsquigarrow (a \to c)] = (a \rightsquigarrow b) \rightsquigarrow [(b \rightsquigarrow c) \to (a \rightsquigarrow c)] = 1$;
(iv) $1^- = 1^\sim$;
(v) $(a^- \rightsquigarrow b^-) \to (b \to a) = (a^\sim \to b^\sim) \to (b \rightsquigarrow a) = 1$;
(vi) $(a \to b^-)^\sim = (b \rightsquigarrow a^\sim)^-$.
For details about pseudo-Wajsberg algebras see [9, 10].

Proposition 4.4 The variety of bounded Wajsberg pseudo-hoops is termwise definitionally equivalent to the variety of pseudo-Wajsberg algebras.

Proof: If $\mathbf{A} = (A, \odot, \to, \rightsquigarrow, 0, 1)$ is a bounded Wajsberg pseudo-hoop, then for all $a \in A$ we define $a^- = a \to 0$ and $a^\sim = a \rightsquigarrow 0$. Then, algebra $\mathbf{A}^* = (A, \to, \rightsquigarrow, ^-, ^\sim, 1)$ is a pseudo-Wajsberg algebra. Conversely, if $\mathbf{B} = (B, \to, \rightsquigarrow, ^-, ^\sim, 1)$

is a pseudo-Wajsberg algebra, then let $0 = 1^- = 1^\sim$ and for all $a, b \in A$, $a \odot b = (a \to b^-)^\sim = (b \rightsquigarrow a^\sim)^-$. Then $\mathbf{B}^\circ = (B, \odot, \to, \rightsquigarrow, 0, 1)$ is a bounded Wajsberg pseudo-hoop. It is easy to prove that for any bounded Wajsberg pseudo-hoop $\mathbf{A}$ and for any pseudo-Wajsberg algebra $\mathbf{B}$, we have that $\mathbf{A}^{*\circ} = \mathbf{A}$ and $\mathbf{B}^{\circ *} = \mathbf{B}$. $\Box$

Basic pseudo-hoops

A pseudo-hoop $\mathbf{A} = (A, \odot, \to, \rightsquigarrow, 1)$ is called *basic* if it satisfies the following conditions
(B1) $(a \to b) \to c \leq ((b \to a) \to c) \to c$;
(B2) $(a \rightsquigarrow b) \rightsquigarrow c \leq ((b \rightsquigarrow a) \rightsquigarrow c) \rightsquigarrow c$.

Lemma 4.5 Let $\mathbf{A}$ be a basic pseudo-hoop. For any $a, b, c \in A$, the following hold:
(i) $(a \to b) \vee_1 (b \to a) = 1$;
(i$'$) $(a \rightsquigarrow b) \vee_2 (b \rightsquigarrow a) = 1$;
(ii) $a \to b = (a \vee_1 b) \to b$;
(ii$'$) $a \rightsquigarrow b = (a \vee_2 b) \rightsquigarrow b$;
(iii) $(a \vee_1 b) \to c = (a \to c) \wedge (b \to c)$;
(iii$'$) $(a \vee_2 b) \rightsquigarrow c = (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$.

Proof: (i) Let $x = (a \to b) \vee_1 (b \to a)$. Applying (B1) we get that $(a \to b) \to x \leq ((b \to a) \to x) \to x$. But, by Proposition 2.11(ii), we have that $(a \to b) \to x = (b \to a) \to x = 1$, hence $1 \leq 1 \to x = x$. That is, $x = 1$.
(ii) Since $a \leq a \vee_1 b$, applying (13) we get that $(a \vee_1 b) \to b \leq a \to b$. Let us prove the converse inequality. From (26) and (13) it follows that $a \to b \leq ((a \to b) \rightsquigarrow b) \to b \leq (a \vee_1 b) \to b$, since $a \vee_1 b = ((a \to b) \rightsquigarrow b) \wedge ((b \to a) \rightsquigarrow a) \leq (a \to b) \rightsquigarrow b$.
(iii) The inequality $(a \vee_1 b) \to c \leq (a \to c) \wedge (b \to c)$ is obvious, by (13) and $a, b \leq a \vee_1 b$. Let $x = [(a \to c) \wedge (b \to c)] \rightsquigarrow [(a \vee_1 b) \to c]$. We have to prove that $x = 1$. We have that $[(a \to c) \wedge (b \to c)] \odot [(a \vee_1 b) \to b] \odot (a \vee_1 b) = [(a \to c) \wedge (b \to c)] \odot [(a \vee_1 b) \wedge b] = [(a \to c) \wedge (b \to c)] \odot b \leq (b \to c) \odot b = b \wedge c \leq c$, so $[(a \to c) \wedge (b \to c)] \odot [(a \vee_1 b) \to b] \leq (a \vee_1 b) \to c$, hence $(a \vee_1 b) \to b \leq x$. Applying now (ii), it follows that $a \to b \leq x$, that is $(a \to b) \to x = 1$. We obtain similarly that $(b \to a) \to x = 1$. By (B1), we get that $1 = (a \to b) \to x \leq ((b \to a) \to x) \to x = 1 \to x = x$, hence $x = 1$.
(i$'$), (ii$'$) and (iii$'$) are proved similarly. $\Box$

Proposition 4.6 Let $\mathbf{A}$ be a basic pseudo-hoop. Then for any $a, b \in A$ there exists $a \vee b$ and $a \vee b = a \vee_1 b = a \vee_2 b$. The lattice $(A, \wedge, \vee)$ is distributive.

Proof: By Proposition 2.11(ii), we have that $a, b \leq a \vee_1 b$ and $a, b \leq a \vee_2 b$. Let $c \in A$ such that $a, b \leq c$, that is $a \to c = b \to c = 1$. Applying Lemma 4.5(iii), it follows that $(a \vee_1 b) \to c = (a \to c) \wedge (b \to c) = 1 \wedge 1 = 1$, hence $a \vee_1 b \leq c$.

Similarly, applying Lemma 4.5(iii$'$), we obtain that $a \vee_2 b \leq c$. Thus, we have proved that $a \vee b = a \vee_1 b = a \vee_2 b$. To get that $(A, \wedge, \vee)$ is a distributive lattice apply Lemma 2.9(iii). $\Box$

Proposition 4.7 Let $\mathbf{A}$ be a pseudo-hoop. The following are equivalent:
(i) $\mathbf{A}$ is a basic pseudo-hoop;
(ii) $\vee_1$ and $\vee_2$ are associative and $(a \to b) \vee_1 (b \to a) = 1$ for all $a, b \in A$;
(iii) $\vee_1$ and $\vee_2$ are associative and $(a \rightsquigarrow b) \vee_2 (b \rightsquigarrow a) = 1$ for all $a, b \in A$.

Proof: (i)$\Rightarrow$(ii) By Proposition 4.6 and Lemma 4.5(i).
(ii)$\Rightarrow$(i) By Remark 2.14, we have that $\vee = \vee_1 = \vee_2$. Applying (11) and Proposition 2.8(i), it follows that $((a \to b) \to c) \odot ((b \to a) \to c) \leq ((a \to b) \to c) \wedge ((b \to a) \to c) = ((a \to b) \vee (b \to a)) \to c = 1 \to c = c$. Hence, $((a \to b) \to c) \leq ((b \to a) \to c) \to c$, that is (B1). We prove similarly (B2).
(i)$\Leftrightarrow$(iii) Similarly. $\Box$

Lemma 4.8 Let $\mathbf{A}$ be a basic pseudo-hoop. Then for all $a, b, c \in A$,
(i) $a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c)$;
(i$'$) $(b \wedge c) \odot a = (b \wedge a) \odot (c \wedge a)$;
(ii) $(a \to b) \to (b \to a) = b \to a$;
(ii$'$) $(a \rightsquigarrow b) \rightsquigarrow (b \rightsquigarrow a) = b \rightsquigarrow a$.

Proof: (i) Applying the fact that $(A, \wedge, \vee)$ is distributive, Lemma 4.5(i$'$) and Lemma 2.9(i$'$), we get that $(a \odot b) \wedge (a \odot c) = [(a \odot b) \wedge (a \odot c)] \odot 1 = [(a \odot b) \wedge (a \odot c)] \odot [(b \rightsquigarrow c) \vee (c \rightsquigarrow b)] = [((a \odot b) \wedge (a \odot c)) \odot (b \rightsquigarrow c)] \vee [((a \odot b) \wedge (a \odot c)) \odot (c \rightsquigarrow b)] \leq [a \odot b \odot (b \rightsquigarrow c)] \vee [a \odot c \odot (c \rightsquigarrow b)] = a \odot (b \wedge c)$. The converse inequality is obvious.
(i$'$) Similarly.
(ii) By (9), we get that $b \to a \leq (a \to b) \to (b \to a)$. Conversely, we have that $1 = (b \to a) \vee (a \to b) = [((b \to a) \to (a \to b)) \rightsquigarrow (a \to b)] \wedge [((a \to b) \to (b \to a)) \rightsquigarrow (b \to a)]$, hence $((a \to b) \to (b \to a)) \rightsquigarrow (b \to a) = 1$, that is $(a \to b) \to (b \to a) \leq b \to a$.
(ii$'$) Similarly. $\Box$

Proposition 4.9 Any Wajsberg pseudo-hoop is a basic pseudo-hoop.

Proof: Let $a, b \in A$. By (32), (W1) and (30), it follows that $b \to a = [((b \to a) \rightsquigarrow a) \to b] \to (b \to a) = [((a \to b) \rightsquigarrow b) \to b] \to (b \to a) = (a \to b) \to (b \to a)$. By Proposition 4.3(i), $(a \to b) \vee_1 (b \to a) = ((a \to b) \to (b \to a)) \rightsquigarrow (b \to a) = (b \to a) \rightsquigarrow (b \to a) = 1$. Since $\mathbf{A}$ is Wajsberg, by Proposition 4.3(ii) we get also that $\vee_1$ and $\vee_2$ are associative. Applying now Proposition 4.7, we obtain that $\mathbf{A}$ is a basic pseudo-hoop. $\Box$

Bounded basic hoops are termwise definitionally equivalent to BL-algebras (see [1], Theorem 2.6). A similar result is obtained in the case of pseudo-hoops.
A *pseudo-BL algebra* ([12, 20]) is an algebra $(A, \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ with five

binary operations $\vee, \wedge, \odot, \rightarrow, \rightsquigarrow$ and two constants $0, 1$ such that:
(i) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;
(ii) $(A, \odot, 1)$ is a monoid;
(iii) $a \odot b \leq c$ iff $a \leq b \rightarrow c$ iff $b \leq a \rightsquigarrow c$;
(iv) $a \wedge b = (a \rightarrow b) \odot a = a \odot (a \rightsquigarrow b)$;
(v) $(a \rightarrow b) \vee (b \rightarrow a) = (a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$.

Proposition 4.10 The variety of bounded basic pseudo-hoops is termwise definitionally equivalent to the variety of pseudo-BL algebras.

Proof: If $\mathbf{A} = (A, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a bounded basic pseudo-hoop, then for all $a, b \in A$ we can define $a \wedge b$ and $a \vee b$. Then, algebra $\mathbf{A}^* = (A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a pseudo-BL algebra. Conversely, if $\mathbf{B} = (B, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a pseudo-BL algebra, then its $\{\odot, \rightarrow, \rightsquigarrow, 0, 1\}$-reduct is a bounded basic pseudo-hoop $\mathbf{B}^\circ$. $\square$

Product pseudo-hoops

A basic pseudo-hoop $\mathbf{A} = (A, \odot, \rightarrow, \rightsquigarrow, 1)$ is called *product* if it satisfies the following conditions
(P1) $b \rightarrow b^2 \leq (a \wedge (a \rightarrow b)) \rightarrow b$;
(P2) $b \rightsquigarrow b^2 \leq (a \wedge (b \rightsquigarrow a)) \rightarrow b$;
(P3) $((a \rightarrow b) \rightarrow b) \odot (c \odot a \rightarrow d \odot a) \odot (c \odot b \rightarrow d \odot b) \leq c \rightarrow d$;
(P4) $((a \rightsquigarrow b) \rightsquigarrow b) \odot (a \odot c \rightsquigarrow a \odot d) \odot (b \odot c \rightsquigarrow b \odot d) \leq c \rightsquigarrow d$.

Proposition 4.11 If $\mathbf{A}$ is a cancellative and a basic pseudo-hoop, then $\mathbf{A}$ is a product pseudo-hoop.

Proof: Let $a, b, c, d \in A$.
(P1) $b \rightarrow b^2 = 1 \odot b \rightarrow b \odot b = 1 \rightarrow b = b \leq (a \wedge (a \rightarrow b)) \rightarrow b$, by Proposition 4.2(i) and (9).
(P2) Similarly, applying Proposition 4.2(i$'$) and (9).
(P3) $((a \rightarrow b) \rightarrow b) \odot (c \odot a \rightarrow d \odot a) \odot (c \odot b \rightarrow d \odot b) = ((a \rightarrow b) \rightarrow b) \odot (c \rightarrow d) \odot (c \rightarrow d) \leq c \rightarrow d$.
(P4) Similarly. $\square$

(Strongly) simple pseudo-hoops

A pseudo-hoop $\mathbf{A}$ is called *simple* if $\{1\}$ is the unique proper normal filter of $\mathbf{A}$. The pseudo-hoop $\mathbf{A}$ is called *strongly simple* if $\{1\}$ is the unique proper filter of $\mathbf{A}$. Of course, any strongly simple pseudo-hoop is simple. When $\mathbf{A}$ is a hoop the two notions coincide, since in this case filters and normal filters coincide.

Lemma 4.12 Let $\mathbf{A}$ be a a pseudo-hoop. The following are equivalent:
(i) $\mathbf{A}$ is strongly simple;
(ii) for all $a \in A$, if $a \neq 1$ then $< a >= A$;

(iii) for all $a, b \in A$, if $a \neq 1$ then there exists $n \in \omega - \{0\}$ such that $a^n \leq b$;
(iv) for all $a, b \in A$, if $a \neq 1$ then there exists $n \in \omega - \{0\}$ such that $a \stackrel{n}{\rightarrow} b = 1$;
(iv$'$) for all $a, b \in A$, if $a \neq 1$ then there exists $n \in \omega - \{0\}$ such that $a \stackrel{n}{\rightsquigarrow} b = 1$.

Proof: (i)$\Leftrightarrow$(ii) is obvious.
By Proposition 3.2, any one of the conditions (iii), (iv), and (iv$'$) is equivalent to (ii). $\Box$

Lemma 4.13 Let **A** be a a strongly simple pseudo-hoop. For all $a, b \in A$,
(i) $b \rightarrow a = a$ implies $a = 1$ or $b = 1$;
(i$'$) $b \rightsquigarrow a = a$ implies $a = 1$ or $b = 1$

Proof: (i) Let $a, b \in A$ such that $b \rightarrow a = a$. It follows immediately that $b^n \rightarrow a = a$ for all $n \in \omega$. If $b \neq 1$, then we can apply the above lemma to get $n_0 \in \omega - \{0\}$ such that $b^{n_0} \leq a$, that is $b^{n_0} \rightarrow a = 1$. Hence $a = 1$.
(i$'$) Similarly. $\Box$

Proposition 4.14 Let **A** be a a basic pseudo-hoop such that for all $a, b \in A$,
$b \rightarrow a = a$ implies $a = 1$ or $b = 1$
and
$b \rightsquigarrow a = a$ implies $a = 1$ or $b = 1$.
Then **A** is a linear Wajsberg pseudo-hoop.

Proof: Let $a, b \in A$. Applying Lemma 4.8(ii), we get that $(a \rightarrow b) \rightarrow (b \rightarrow a) = (b \rightarrow a)$. Hence, by the hypothesis, $a \rightarrow b = 1$ or $b \rightarrow a = 1$, that is $a \leq b$ or $b \leq a$. Thus, **A** is a linear pseudo-hoop. Let us prove now that **A** is Wajsberg. We shall prove only (W1), (W2) following similarly. Let $a, b \in A$. If $a = b$, then (W1) is obvious. Assume $a \neq b$ Since **A** is linear, we can suppose that $a < b$. It follows that $(a \rightarrow b) \rightsquigarrow b = 1 \rightsquigarrow b = b$, so it suffices to show that $(b \rightarrow a) \rightsquigarrow a = b$.
By (32), $[((b \rightarrow a) \rightsquigarrow a) \rightarrow b] \rightarrow (b \rightarrow a) = b \rightarrow a$, so by the hypothesis and the fact that $b \rightarrow a \neq 1$, it follows that $((b \rightarrow a) \rightsquigarrow a) \rightarrow b = 1$, that is $(b \rightarrow a) \rightsquigarrow a \leq b$. But, from (24), we have also that $b \leq (b \rightarrow a) \rightsquigarrow a$. Hence, we have obtained that $(b \rightarrow a) \rightsquigarrow a = b$. $\Box$

Corollary 4.15 Every strongly simple basic pseudo-hoop is a linear Wajsberg pseudo-hoop.

Proof: It follows immediately from Lemma 4.13 and Proposition 4.14 $\Box$

5 Some examples

In this section we shall give some examples of pseudo-hoops and normal filters, inspired by [18, 19, 12, 3, 1].

Example 5.1 (see [18], Example 1.1, and [19], Example 1.3)
Let $\mathbf{G} = (G, +, -, 0, \vee, \wedge)$ be an arbitrary l-group. For an arbitrary element $u \in G$, $u \geq 0$ define on the set $G[u] = [0, u]$ the following operations:
$a \odot b = (a - u + b) \vee 0$,
$a \to b = (b - a + u) \wedge u$, and
$a \rightsquigarrow b = (u - a + b) \wedge u$.
By [19], Proposition 1.4, $\mathbf{G[u]} = (G[u], \odot, \to, \rightsquigarrow, u)$ is a pseudo-MV algebra, hence a bounded Wajsberg pseudo-hoop. For the sake of completeness, we shall give here a proof of this fact. Firstly, let us note that $G[u]$ is closed under the operations $\odot, \to, \rightsquigarrow$. We shall verify the identities from Theorem 2.2. Let $a, b, c \in [0, u]$.
(A0) $a \odot u = (a - u + u) \vee 0 = a \vee 0 = a$ and $u \odot a = (u - u + a) \vee 0 = a \vee 0 = a$, since $a \geq 0$;
(A1) $a \to a = a \rightsquigarrow a = u \wedge u = u$;
(A2) $a \odot b \to c = [(a-u+b) \vee 0] \to c = [c-(a-u+b) \vee 0+u] \wedge u = (c-b+u-a+u) \wedge (c+u) \wedge u = (c-b+u-a+u) \wedge u$ and $a \to (b \to c) = [(b \to c) - a + u] \wedge u = [(c-b+u) \wedge u - a + u] \wedge u = (c-b+u-a+u) \wedge (u-a+u) \wedge u = (c-b+u-a+u) \wedge u$;
(A3) $b \odot a \rightsquigarrow c = [(b-u+a) \vee 0] \rightsquigarrow c = [u-(b-u+a) \vee 0+c] \wedge u = (u-a+u-b+c) \wedge (u+c) \wedge u = (u-a+u-b+c) \wedge u$ and $a \rightsquigarrow (b \rightsquigarrow c) = a \rightsquigarrow [(u-b+c) \wedge u] = [u-a+(u-b+c) \wedge u] \wedge u = (u-a+u-b+c) \wedge (u-a+u) \wedge u = (u-a+u-b+c) \wedge u$;
(A4) $(a \to b) \odot a = [(a \to b) - u + a] \vee 0 = [(b - a + u) \wedge u - u + a] \vee 0 = [(b - a + u - u + a) \wedge (u - u + a)] \vee 0 = (b \wedge a) \vee 0 = b \wedge a$ and, similarly, $(b \to a) \odot b = a \wedge b$. We also have that $a \odot (a \rightsquigarrow b) = [a - u + (a \rightsquigarrow b)] \vee 0 = [a-u+(u-a+b) \wedge u] \vee 0 = [(a-u+u-a+b) \wedge (a-u+u)] \vee 0 = (b \wedge a) \vee 0 = b \wedge a$ and, similarly, $b \odot (b \rightsquigarrow a) = a \wedge b$.
Hence, $\mathbf{G[u]}$ is a pseudo-hoop. It is obvious that $\mathbf{G[u]}$ is bounded. It remains to prove (W1) and (W2). Let $a, b \in G[u]$. We have that $(a \to b) \rightsquigarrow b = [u - (a \to b) + b] \wedge u = [u-(b-a+u) \wedge u+b] \wedge u = [(u-u+a-b+b) \vee (u-u+b)] \wedge u = (a \vee b) \wedge u = a \vee b$ and $(b \to a) \rightsquigarrow a = [u - (b \to a) + a] \wedge u = [u - (a - b + u) \wedge u + a] \wedge u = [(u - u + b - a + a) \vee (u - u + a)] \wedge u = (b \vee a) \wedge u = b \vee a = a \vee b$. Hence, (W1) is satisfied. We prove similarly that (W2) holds.

Let K be a normal convex l-subgroup of G. We define
$F = \{a \in G[u] \mid u - a \in K\}$.

Proposition 5.2 F is a normal filter of $\mathbf{G[u]}$.

Proof: Firstly, let us remark that, since K is normal, if $u - a \in K$, then $u = (u - a) + a \in K + a = a + K$, hence $-a + u \in K$. That is, $F = \{a \in G[u] \mid -a + u \in K\}$. We have that $u \in F$, since $u - u = 0 \in K$. Let $a \in F, b \in G[u]$ such that $a \leq b$. Since K is convex, $0 \leq u-b \leq u-a$, and $0, u-a \in K$ it follows that $u-b \in K$, hence $b \in F$. Let $a, b \in F$, that is $u-a, u-b \in K$. We get that $0 \leq u-(a \odot b) = u-[(a-u+b) \vee 0] = (u-b+u-a) \wedge u \leq (u-b)+(u-a) \in K$, so $u-(a \odot b) \in K$, hence $a \odot b \in F$. Thus, we have got that F is a filter of $\mathbf{G[u]}$.

In order to prove that F is normal we apply Proposition 3.10(ii). We shall prove only that $x \odot F \subseteq F \odot x$ for any $x \in G[u]$, the converse inclusion being similar. Let $a \in F$, that is $-a+u \in K$, so $-u+a = -(-a+u) \in K$. Since K is normal, we get that $x-u+a \in x+K = K+x$, hence $x-u+a = k+x$ for some $k \in K$. It follows that $k = x-u+a-x$, so $k+u = x-u+a-x+u$. Let $b = [(k+u)\vee 0]\wedge u$. Then, $0 \leq b \leq u$ and $0 \leq u-b = [u-(k+u)\vee 0]\vee 0 = [(u-u-k)\wedge u]\vee 0 = [(-k)\wedge u]\vee 0 = [(-k)\vee 0]\wedge u \leq (-k)\vee 0 = |k| \in K$, so $u-b \in K$, hence $b \in F$. We get that $b \odot x = (b-u+x)\vee 0 = [((k+u)\vee 0)\wedge u - u + x]\vee 0 = [((k+u)\vee 0 - u + x)\wedge (u-u+x)]\vee 0 = [((k+u-u+x)\vee(-u+x))\wedge x]\vee 0 = [((k+x)\vee(-u+x))\wedge x]\vee 0 = [(k+x)\vee(-u+x)\vee 0]\wedge x = [(k+x)\vee 0]\wedge x = [(x-a+u)\vee 0]\wedge x = (x\odot a)\wedge x = x \odot a$. Hence, $a \odot x = x \odot b \in x \odot F$. □

The following example is strongly related to Example 1.21 from [12].

Example 5.3 Let $\mathbf{G} = (G, +, -, 0, \vee, \wedge)$ be an arbitrary l-group and $N(G)$ be the negative cone of $\mathbf{G}$, that is $N(G) = \{a \in G \mid a \leq 0\}$. On $N(G)$ we define the following operations:

$a \odot b = a + b$,

$a \to b = (b-a)\wedge 0$, and

$a \rightsquigarrow b = (-a+b)\wedge 0$.

Then $\mathbf{N(G)} = (N(G), \odot, \to, \rightsquigarrow, 0)$ is a pseudo-hoop. We shall verify the identities from Theorem 2.2. Let $a, b, c \leq 0$.

(A0) $a \odot 0 = a + 0 = a = 0 + a = 0 \odot a$;

(A1) $a \to a = a \rightsquigarrow a = 0 \wedge 0 = 0$;

(A2) $a \odot b \to c = [c-(a+b)]\wedge 0 = (c-b-a)\wedge 0$ and $a \to (b \to c) = [(c-b)\wedge 0 - a]\wedge 0 = (c-b-a)\wedge(-a)\wedge 0 = (c-b-a)\wedge 0$;

(A3) $b \odot a \rightsquigarrow c = [-(b+a)+c]\wedge 0 = (-a-b+c)\wedge 0$ and $a \rightsquigarrow (b \rightsquigarrow c) = [-a+(-b+c)\wedge 0]\wedge 0 = (-a-b+c)\wedge(-a)\wedge 0 = (-a-b+c)\wedge \wedge 0$;

(A4) $(a \to b)\odot a = (b-a)\wedge 0 + a = b \wedge a$ and, similarly, $(b \to a)\odot b = a \wedge b$. $a \odot (a \rightsquigarrow b) = a + (-a+b)\wedge 0 = b \wedge a$ and, similarly, $b \odot (b \rightsquigarrow a) = a \wedge b$.

Proposition 5.4 The pseudo-hoop $\mathbf{N(G)}$ is cancellative.

Proof: Let us verify (C1) and (C2). If $a, b \in G[u]$, then $b \to a \odot b = (a+b-b)\wedge 0 = a \wedge 0 = a$ and $b \rightsquigarrow b \odot a = (-b+b+a)\wedge 0 = a \wedge 0 = a$. □

Proposition 5.5 The pseudo-hoop $\mathbf{N(G)}$ is a product pseudo-hoop.

Proof: Let $a, b, c \in A$. Firstly, we prove that $\mathbf{N(G)}$ is basic. We shall verify only (B1), the proof of (B2) being similar. We have that $(a \to b) \to c = [c-(b-a)\wedge 0]\wedge 0 = [(c-a+b)\vee c]\wedge 0$, $(b \to a) \to c = [(c-b+a)\vee c]\wedge 0$, and $((b \to a) \to c) \to c = [c-(((c-b+a)\vee c)\wedge 0)]\wedge 0 = [(c-(c-b+a)\vee c)\vee c]\wedge 0 = [((c-a+b-c)\wedge 0)\vee c]\wedge 0 = [(c-a+b-c)\vee c]\wedge(0\vee c)\wedge 0 = [(c-a+b-c)\vee c]\wedge 0$. Since $c \leq 0$, it follows that $0 \leq -c$, so $c-a+b \leq c-a+b-c$. That is, $(a \to b) \to c = [(c-a+b)\vee c]\wedge 0 \leq [(c-a+b-c)\vee c]\wedge 0 = ((b \to a) \to c) \to c$.

Apply now Proposition 5.4 and Proposition 4.11 to get that $\mathbf{N}(\mathbf{G})$ is a product pseudo-hoop. $\Box$

Proposition 5.6 (i) If K is a convex l-subgroup of $\mathbf{G}$, then $F = K \cap N(G)$ is a filter of $\mathbf{N}(\mathbf{G})$.
(ii) If K is normal in $\mathbf{G}$, then F is normal in $\mathbf{N}(\mathbf{G})$.

Proof: Obvious.

Proposition 5.7 For any cancellative pseudo-hoop $\mathbf{A}$ there exists an l-group $\mathbf{G}$ such that $\mathbf{A} \cong \mathbf{N}(\mathbf{G})$

Proof: Applying Nakada Theorem (see [17], Theorem X.1), there is an l-group $\mathbf{G}$ and a p.o.-monoid isomorphism $f : (A, \odot, 1, \leq) \to (N(G), +, 0, \leq)$. For all $a, b \in A$, $[f(a) \to f(b)] + f(a) = [(f(b) - f(a)) \wedge 0] + f(a) = f(b) \wedge f(a) = f(b \wedge a) = f((a \to b) \odot a) = f(a \to b) + f(a)$. Since $\mathbf{N}(\mathbf{G})$ is cancellative, it follows that $f(a) \to f(b) = f(a \to b)$. We prove similarly that $f(a) \rightsquigarrow f(b) = f(a \rightsquigarrow b)$. Hence, f is a pseudo-hoop isomorphism. $\Box$

Example 5.8 (see [3], pag. 18, and [1], pag. 5)
Let $\mathbf{A}$, $\mathbf{B}$ be pseudo-hoops with $A \cap B = \{1\}$. Then, the *ordinal sum* of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \times \mathbf{B}$ and is defined as follows. The domain of the algebra $\mathbf{A} \times \mathbf{B}$ is $A \cup B$, $1^{\mathbf{A} \times \mathbf{B}} = 1$,

$$x \odot y = \begin{cases} x \odot^{\mathbf{A}} y & \text{if } x, y \in A \\ x \odot^{\mathbf{B}} y & \text{if } x, y \in B \\ y & \text{if } x \in B - \{1\}, y \in A - \{1\} \\ x & \text{if } x \in A - \{1\}, y \in B - \{1\} \end{cases}$$

$$x \to y = \begin{cases} x \to^{\mathbf{A}} y & \text{if } x, y \in A \\ x \to^{\mathbf{B}} y & \text{if } x, y \in B \\ y & \text{if } x \in B - \{1\}, y \in A - \{1\} \\ 1 & \text{if } x \in A - \{1\}, y \in B - \{1\} \end{cases}$$

$$x \rightsquigarrow y = \begin{cases} x \rightsquigarrow^{\mathbf{A}} y & \text{if } x, y \in A \\ x \rightsquigarrow^{\mathbf{B}} y & \text{if } x, y \in B \\ y & \text{if } x \in B - \{1\}, y \in A - \{1\} \\ 1 & \text{if } x \in A - \{1\}, y \in B - \{1\} \end{cases}$$

If $A \cap B \neq \{1\}$, then we can replace $\mathbf{A}$ and $\mathbf{B}$ with isomorphic copies whose intersection is $\{1\}$. It is enough to take $\mathbf{A} \times \{1^{\mathbf{B}}\}$ and $\mathbf{B} \times \{1^{\mathbf{A}}\}$ and define their ordinal sum as above.

Proposition 5.9 Let $\mathbf{A}$, $\mathbf{B}$ be pseudo-hoops with $A \cap B = \{1\}$. Then
(i) $\mathbf{A} \times \mathbf{B}$ is a pseudo-hoop;
(ii) if $\mathbf{A}$, $\mathbf{B}$ are linear pseudo-hoops, then $\mathbf{A} \times \mathbf{B}$ is also linear;
(iii) $\mathbf{A}$ and $\mathbf{B}$ are subalgebras of $\mathbf{A} \times \mathbf{B}$;
(iv) B is a normal filter of $\mathbf{A} \times \mathbf{B}$.

Proof: (i) The proof is similar to the one given for hoops (see [11]).
(ii) Let $x, y \in A \cup B$. If $x, y \in A$ or $x, y \in B$, then use the fact that **A**, **B** are linear to get that $x \le y$ or $y \le x$. Suppose that $x \in A - \{1\}$ and $y \in B - \{1\}$. Then $x \to y = x \rightsquigarrow y = 1$, so $x \le y$. Similarly, if $x \in B - \{1\}$ and $y \in A - \{1\}$, then $y \to x = y \rightsquigarrow x = 1$, so $y \le x$. Thus, we have proved that $\mathbf{A} \times \mathbf{B}$ is a linear pseudo-hoop.
(iii) is obvious.
(iv) It is easy to prove that B is a filter of $\mathbf{A} \times \mathbf{B}$. Let us prove that B is normal. Let $x, y \in A \cup B$. If $x, y \in B$, then $x \to y = x \to^{\mathbf{B}} y \in B$ and $x \rightsquigarrow y = x \rightsquigarrow^{\mathbf{B}} y \in B$. If $x, y \in A$, then $x \to y = x \to^{\mathbf{A}} y \in A$ and $x \rightsquigarrow y = x \rightsquigarrow^{\mathbf{A}} y \in A$. Since $A \cap B = \{1\}$, we have that $x \to y \in B$ iff $x \to y = 1$ iff $x \le y$ iff $x \rightsquigarrow y = 1$ iff $x \rightsquigarrow y \in B$. If $x \in A - \{1\}, y \in B - \{1\}$, then $x \to y = 1 = x \rightsquigarrow y$. Finally, if $x \in B - \{1\}, y \in A - \{1\}$, then $x \to y = y = x \rightsquigarrow y$. □

Let us consider the two element Boolean algebra $\mathbf{L}_2 = \{0, 1\}$. Then $\mathbf{L}_2$ is a hoop that is not cancellative, since $0 \odot 1 = 0 \odot 0 = 0$ and $1 \ne 0$. If **A** is a pseudo-hoop that is not linear (a direct product of pseudo-hooops, e.g.), then let $\mathbf{B} = \mathbf{A} \times \mathbf{L}_2$.

Remark 5.10 **B** is not a a basic pseudo-hoop.

Proof: Let $a, b \in A$ such that a, b are incomparable, so $a \to_{\mathbf{A}} b \ne 1$ and $b \to_{\mathbf{A}} a \ne 1$. Then $(a \to b) \rightsquigarrow 0 = (a \to_{\mathbf{A}} b) \to 0 = 1$ and $((b \to a) \to 0) \to 0 = 1 \to 0 = 0$, hence (B1) is not satisfied. □

By Proposition 4.9, it follows also that **B** is not a Wajsberg pseudo-hoop. Hence, as in the case of hoops (see [1], pag. 12), the ordinal sum construction allows us to obtain examples of pseudo-hoops that are not basic.

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