Compact representations of BL-algebras

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Abstract

In this paper we define sheaf spaces of BL-algebras (or BL-sheaf spaces), we study completely regular and compact BL-sheaf spaces and compact representations of BL-algebras and, finally, we prove that the category of non-trivial BL-algebras is equivalent with the category of compact local BL-sheaf spaces.

1 Introduction

BL-algebras are the algebraic structures for Hájek's Basic Logic [13]. The main example of a BL-algebra is the interval [0,1] endowed with the structure induced by a continuous t-norm.

In this paper we study compact representations of BL-algebras, following techniques used for ringed spaces by Mulvey [17, 16, 18]. In [16], Mulvey extended the concepts of complete regularity and compactness from topological spaces to ringed spaces and proved a compactness theorem for completely regular ringed spaces generalizing the Gelfand-Kolmogoroff criterion concerning maximal ideals in the ring $\mathbf{R}(X)$ of continuous real functions on a completely regular space X [10]. In [17], Mulvey introduced compact representations of rings, showing that they are exactly those representations of rings that establish an equivalence of categories of modules. Using compact representations, Mulvey extended the Gelfand duality between the categories of compact spaces and commutative C^*-algebras to Gelfand rings [18].

Gelfand rings are characterized by a property that can be formulated in terms of universal algebra, namely that each prime ideal is contained in a unique maximal

ideal. Universal algebras with this property and their Gelfand representations were studied by Georgescu and Voiculescu [12] and, in a lattice-theoretical setting, by Simmons [21].
MV-algebras [3], lattice-ordered groups [1], and BL-algebras are classes of algebras that also satisfy this property. Hence, the problem of obtaining similar results for these structures is natural. Some sheaf representations for lattice-ordered groups are studied in [14]. Filipoiu and Georgescu [9] proved that the category of MV-algebras is equivalent with the category of compact sheaf spaces of MV-algebras with local stalks.
In the present paper, we give an answer for this problem in the case of BL-algebras. In different classes of problems, sheaf representations of universal algebras are very useful since they reduce the study of algebras to the study of the stalks, which usually have a better known structure. In the case of our compact representations, the stalks are local BL-algebras, introduced and studied by Turunen and Sessa [25].
In the first section of the paper we recall some facts about BL-algebras and we study some special filters of BL-algebras, used in the sequel. In Section 2, we define sheaf spaces of BL-algebras (or BL-sheaf spaces), BL-algebras of global sections, morphisms of BL-sheaf spaces and other notions related with sheaf theory.
In the next section we define and study completely regular and compact BL-sheaf spaces and we prove the compactness theorem.
In the following section we remind some general results concerning sheaf representations of BL-algebras and we study a special kind of representations, namely compact representations. We prove that any compact representation of a BL-algebra arises canonically from a family of filters of the BL-algebra satisfying certain conditions.
Finally, in the last section of the paper we prove that the functor from the category of compact local BL-sheaf spaces to the category of non-trivial BL-algebras, obtained by assigning to each BL-sheaf space the BL-algebra of global sections determines an equivalence between these categories.

2 BL-algebras. Definitions and first properties

A *BL-algebra* [13] is an algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, \odot, \rightarrow$ and two constants $0, 1$ such that:
(i) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
(ii) $(A, \odot, 1)$ is a commutative monoid;
(iii) $\odot$ and $\rightarrow$ form an adjoint pair, i.e.
$c \leq a \rightarrow b$ iff $a \odot c \leq b$ for all $a, b, c \in A$;
(iv) $a \wedge b = a \odot (a \rightarrow b)$;
(v) $(a \rightarrow b) \vee (b \rightarrow a) = 1$.
A BL-algebra A is nontrivial iff $0 \neq 1$.
For any BL-algebra A, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice.

A *BL-chain* is a linear BL-algebra, i.e. a BL-algebra such that its lattice order is total.
For any $a \in A$, we define $a^- = a \to 0$. We denote the set of natural numbers by ω. We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \in \omega - \{0\}$. The *order* of $a \in A$, in symbols $ord(a)$, is the smallest $n \in \omega$ such that $a^n = 0$. If no such n exists, then $ord(a) = \infty$.
The following properties hold in any BL-algebra A and will be used in the sequel:
(1.1) $a \odot b \leq a \wedge b \leq a, b$
(1.2) $a \leq b$ implies $a \odot c \leq b \odot c$
(1.3) $0 \to a = 1$ and $1 \to a = a$
(1.4) $a \to b = 1$ iff $a \leq b$
(1.5) $a \odot b = 0$ iff $a \leq b^-$
(1.6) $a \odot a^- = 0$
(1.7) $a \to (b \to c) = (a \odot b) \to c$
(1.8) $(a \wedge b)^- = a^- \vee b^-$ and $(a \vee b)^- = a^- \wedge b^-$
(1.9) $1^- = 0$ and $0^- = 1$
(1.10) $a^- = 1$ iff $a = 0$
(1.11) $a \vee b = 1$ implies $a^n \vee b^n = 1$ for any $n \in \omega$

Let A be a BL-algebra. A *filter* of A is a nonempty set $F \subseteq A$ such that for all $a, b \in A$,
(i) $a, b \in F$ implies $a \odot b \in F$;
(ii) $a \in F$ and $a \leq b$ imply $b \in F$.
A filter F of A is *proper* iff $F \neq A$.
By (1.1) it is obvious that any filter of A is also a filter of the lattice $L(A)$. A proper filter P of A is called *prime* provided that it is prime as a filter of $L(A)$:
$a \vee b \in P$ implies $a \in P$ or $b \in P$.
A proper filter M of A is called *maximal* (or *ultrafilter*) if it is not contained in any other proper filter.
We shall denote by $Spec(A)$ the set of prime filters of A and by $Max(A)$ the set of maximal filters of A. Let us remind some properties of filters that will be used in the sequel.

Proposition 2.1. [24, Proposition 8]
If A is a nontrivial BL-algebra, then any proper filter of A can be extended to a maximal filter.

Proposition 2.2. [24, Proposition 6]
Let P be a prime filter of a nontrivial BL-algebra A. Then the set
$\mathcal{F} = \{F \mid P \subseteq F$ *and* F *is a proper filter of* $A\}$
is linearly ordered with respect to set-theoretical inclusion.

Proposition 2.3. [8, Proposition 1.6]
If A is a nontrivial BL-algebra, then any prime filter of A is contained in a unique maximal filter.

Proposition 2.4. [24, Proposition 7]
Any maximal filter of A is a prime filter of A.

Let $X \subseteq A$. The filter of A generated by X will be denoted by $< X >$. We have that $< \emptyset >= \{1\}$ and $< X >= \{a \in A \mid x_1 \odot \cdots \odot x_n \leq a$ for some $n \in \omega - \{0\}$ and some $x_1, \cdots, x_n \in X\}$ if $\emptyset \neq X \subseteq A$. For any $a \in A$, $< a >$ denotes the principal filter of A generated by $\{a\}$. Then, $< a >= \{b \in A \mid a^n \leq b$ for some $n \in \omega - \{0\}\}$.

Lemma 2.5. *Let F, G be filters of A. Then*
$< F \cup G >= \{a \in A \mid b \odot c \leq a$ *for some* $b \in F,\ c \in G\}$

Proposition 2.6. *Let $\mathcal{F}(A)$ be the set of filters of A. Then $(\mathcal{F}(A), \subseteq)$ is a complete lattice. For every family $\{F_i\}_{i \in I}$ of filters of A, we have that*
$\wedge_{i \in I} F_i = \cap_{i \in I} F_i$ *and* $\vee_{i \in I} F_i =< \cup_{i \in I} F_i >$.

With any filter F of A we can associate a congruence relation $\sim_F$ on A by defining
$a \sim_F b$ iff $a \to b \in F$ and $b \to a \in F$ iff $(a \to b) \odot (b \to a) \in F$.
For any $a \in A$, let a/F be the equivalence class $a/\sim_F$. If we denote by A/F the quotient set $A/_{\sim_F}$, then A/F becomes a BL-algebra with the natural operations induced from those of A.

Proposition 2.7. [13]
Let F be a filter of A and $a, b \in A$.
(i) $a/F = 1/F$ iff $a \in F$;
(ii) $a/F = 0/F$ iff $a^- \in F$;
(iii) for all $a, b \in A$,
$a/F \leq b/F$ *iff* $a \to b \in F$;
(iv) A/F is a BL-chain iff F is prime.

If $h : A \to B$ is a BL-morphism, then the *kernel* of h is the set $Ker(h) = \{a \in A \mid h(a) = 1\}$. It is easy to see that

Proposition 2.8. *Let $h : A \to B$ be a BL-morphism. If G is a (proper, prime) filter of B, then $h^{-1}(G)$ is a (proper, prime) filter of A. Thus, in particular, $Ker(h)$ is a proper filter of A.*

Lemma 2.9. [7, Proposition 1.13]
Let A be a nontrivial BL-algebra and M a proper filter of A. The following are equivalent:
(i) M is maximal;
(ii) for any $x \in A$,
$x \notin M$ *implies* $(x^n)^- \in M$ *for some* $n \in \omega$.

Proposition 2.10. *Let $h : A \to B$ be a BL-morphism. If N is a maximal filter of B, then $h^{-1}(N)$ is a maximal filter of A.*

Proof. By Proposition 2.8, we have that $h^{-1}(N)$ is a proper filter of A. In order to get that it is maximal, we shall apply Lemma 2.9. Let $x \in A$ such that $x \notin h^{-1}(N)$, hence $h(x) \notin N$. Since N is a maximal filter of B, there is $n \in \omega$ such that $(h(x)^n)^- \in N$, that is $h((x^n)^-) \in N$, since h is a homomorphism of BL-algebras. We have got that $(x^n)^- \in h^{-1}(N)$. $\square$ $\square$

For any filter F of A, let us denote by $[]_F$ the natural homomorphism from A onto A/F, defined by $[]_F(a) = a/F$ for any $a \in A$. Then $F = Ker([]_F)$.

Proposition 2.11. [11, Proposition 1.12]
Let A be a BL-algebra and F a filter of A.
(i) the map $G \overset{\alpha}{\mapsto} []_F(G)$ is an inclusion-preserving bijective correspondence between the filters of A containing F and the filters of A/F. The inverse map is also inclusion-preserving;
(ii) G is a proper filter of A containing F iff $[]_F(G)$ is a proper filter of A/F. Hence, there is a bijection between the proper filters of A containing F and the proper filters of A/F;
(iii) there is a bijection between the maximal filters of A containing F and the maximal filters of A/F.

Following [25], a BL-algebra A is *local* if it has a unique maximal filter.

Proposition 2.12. [25]
Let A be a local BL-algebra. Then its unique maximal filter is
$\{a \in A \mid ord(a) = \infty\}$.

Proposition 2.13. [25]
Any BL-chain is a local BL-algebra.

Proposition 2.14. *Let P be a proper filter of A. The following are equivalent:*
(i) A/P is a local BL-algebra;
(ii) P is contained in a unique maximal filter of A.

Proof. Apply [11, Proposition 2.6], and [11, Proposition 2.8]. □ □

Let A be a nontrivial BL-algebra. The *prime spectrum* of A is the set $Spec(A)$ of prime filters of A, endowed with the Zariski topology, of which the subsets of the form
$D(a) = \{P \in Spec(A) \mid a \notin P\}$ for $a \in A$
form a basis of open sets.
The *maximal spectrum* of A is the subspace $Max(A)$ of $Spec(A)$ consisting of the maximal filters of A with the induced topology. The subsets
$d(a) = D(a) \cap Max(A) = \{M \in Max(A) \mid a \notin M\}$, $a \in A$
form a basis for the topology of the maximal spectrum. Then $Spec(A)$ is a compact topological space and $Max(A)$ is compact and Hausdorff [15].

In the sequel, we shall remind some facts concerning the reticulation of a BL-algebra A. For details see [15].
Let us define a binary relation $\equiv$ on A by
$a \equiv b$ iff $D(a) = D(b)$.
Then $\equiv$ is an equivalence relation on A compatible with the operations $\odot, \wedge$ and $\vee$. For $a \in A$ let us denote by $[a]$ the class of $a \in A$ with respect to $\equiv$. The bounded distributive lattice $\beta(A) = (A/\equiv, \vee, \wedge, [0], [1])$ is called the *reticulation* of the BL-algebra A.
If $h : A \to B$ is a homomorphism of BL-algebras, then $\beta(h) : \beta(A) \to \beta(B)$,

defined by $\beta(h)([a]) = [h(a)]$, is a homomorphism of bounded distributive lattices. It follows that we can define a functor β from the category of nontrivial BL-algebras to the category of bounded distributive lattices. The functor β is called the *reticulation functor*.
If F is a filter of A, then $\beta(F) = \{[a] \mid a \in A\}$ is a filter of the lattice $\beta(A)$ and the mapping $F \mapsto \beta(F)$ is an isomorphism between the lattice $\mathcal{F}(A)$ of filters of A and the lattice $\mathcal{F}(\beta(A))$ of filters of $\beta(A)$. If $P \in Spec(A)$, then $\beta(P)$ is a prime filter of $\beta(A)$ and the mapping $P \mapsto \beta(P)$ is a homeomorphism between $Spec(A)$ and $Spec(\beta(A))$. Similarly, $Max(A)$ is homeomorphic to $Max(\beta(A))$.

Let us remind that a bounded distributive lattice L is called *normal* [26, 4] if each prime ideal of L contains a unique minimal prime ideal.

Proposition 2.15. [15, Proposition 3.14]
For any nontrivial BL-algebra A, $\beta(A)$ is a normal lattice.

To any prime filter P of a bounded distributive lattice or a BL-algebra A we associate the set
$O(P) = \{a \in A \mid a \vee b = 1 \text{ for some } b \notin P\}$.
Then it is easy to see that $O(P)$ is a proper filter of A such that $O(P) \subseteq P$.
We have the following characterization of normal lattices

Proposition 2.16. [19, Theorem 3]
Let L be a bounded distributive lattice. The following are equivalent:
(i) L is normal;
(ii) for any maximal filter M of L, M is the unique maximal filter that contains $O(M)$.

Lemma 2.17. *For any maximal filter M of A,*
$\beta(O(M)) = O(\beta(M))$.

Proof. In the proof, we use that for all $a \in A$, $[a] = [1]$ iff $a = 1$ and for each maximal filter M of A, $a \in M$ iff $[a] \in \beta(M)$ [15].
"$\subseteq$" Let $[a] \in \beta(O(M))$, so there is $b \in O(M)$ such that $[a] = [b]$. Since $b \in O(M)$, there is $c \notin M$ such that $b \vee c = 1$. It follows that $[a] \vee [c] = [b] \vee [c] = [1]$ and $[c] \notin \beta(M)$. Hence, $[a] \in O(\beta(M))$.
"$\supseteq$" If $[a] \in O(\beta(M))$, then there is $[b] \notin \beta(M)$ such that $[a] \vee [b] = [1]$. Hence, there is $b \notin M$ such that $a \vee b = 1$, that is $a \in O(M)$, so $[a] \in \beta(O(M))$. □ □

Proposition 2.18. *Let A be a nontrivial BL-algebra. Then*
(i) for any maximal filter M of A, M is the unique maximal filter that contains $O(M)$;
(ii) for any distinct maximal filters M, N of A, $O(M) \vee O(N) = A$;
(iii) $A/O(M)$ is local for any $M \in Max(A)$.

Proof. (i) Apply Proposition 2.16, Lemma 2.17 and the properties of the reticulation of A.
(ii) Suppose that $O(M) \vee O(N)$ is a proper filter of A. Then, using Proposition 2.1, we get a contradiction to (i).
(iii) Apply (i) and Proposition 2.14. □ □

Proposition 2.19. [6, Proposition 4.36]
Let A be a nontrivial BL-algebra. Then
$\bigcap_{M \in Max(A)} O(M) = \{1\}$.

3 BL-sheaf spaces. Definitions and first properties

A *sheaf space of BL-algebras* (or a *BL-sheaf space*) is a triple (F, p, X) such that the following properties are satisfied:
(i) F and X are topological spaces;
(ii) $p : F \to X$ is a local homeomorphism from F onto X;
(iii) for each $x \in X$, $p^{-1}(\{x\}) = F_x$ is a nontrivial BL-algebra with operations denoted by $\vee_x, \wedge_x, \odot_x, \to_x, 0_x, 1_x$;
(iv) the functions $(a, b) \mapsto a \vee_x b$, $(a, b) \mapsto a \wedge_x b$, $(a, b) \mapsto a \odot_x b$, $(a, b) \mapsto a \to_x b$ from the set $\{(a, b) \in F \times F \mid p(a) = p(b)\}$ into F are continuous, where $x = p(a) = p(b)$;
(v) the functions $\underline{0}, \underline{1} : X \to F$, which assign to each x in X the zero 0_x and the unit 1_x of F_x respectively, are continuous.

X is known as the *base space*, F as the *total space* and F_x is called the *stalk* of F at $x \in X$.
If $Y \subseteq X$, then a *section* σ over Y is a continuous map $\sigma : Y \to F$ satisfying $(p \circ \sigma)(y) = y$ for all $y \in Y$. The set of all sections over Y form a nontrivial BL-algebra with the operations defined pointwise, that will be denoted by $\Gamma(Y, F)$. The elements of $\Gamma(X, F)$ are called *global sections*.
For every $\sigma, \tau \in \Gamma(Y, F)$, we shall use the following notation:
$[\sigma = \tau] = \{y \in Y \mid \sigma(y) = \tau(y)\}$.
A BL-sheaf space (F, p, X) is called *local* if for each $x \in X$ the stalk F_x is a local BL-algebra.
We shall use the expression *a BL-algebra of global sections* to refer to any BL-subalgebra of $\Gamma(X, F)$. If A is a BL-algebra of global sections, then for each $x \in X$, we define $p_x^A : A \to F_x$ by $p_x^A(\sigma) = \sigma(x)$ for all $\sigma \in A$. If $A = \Gamma(X, F)$, then we shall denote p_x^A by p_x.
The following properties are well-known and will be used in the sequel. For details see [23, 5, 22].

Proposition 3.1. *Let (F, p, X) be a BL-sheaf space.*
(i) for any $Y \subseteq X$ and $\sigma, \tau \in \Gamma(Y, F)$, the subset $[\sigma = \tau]$ is open in Y;
(ii) for each $a \in F$ there are an open subset U of X and a section $\sigma \in \Gamma(U, F)$ such that $p(a) \in U$ and $\sigma(p(a)) = a$;
(iii) if $Z \subseteq Y \subseteq X$ and $\sigma \in \Gamma(Y, F)$, then $\sigma|_Z \in \Gamma(Z, F)$;
(iv) the family $\{\sigma(U) \mid U$ is open in $X, \sigma \in \Gamma(U, F)\}$ is a basis for the topology of F;
(v) if A is a BL-algebra of global sections, then p_x^A is a BL-morphism for each $x \in X$;

(vi) if (F, p, X) and (G, q, X) are BL-sheaf spaces and $f : F \to G$ such that $q \circ f = p$, then

f is continuous iff f is open iff f is a local homeomorphism.

If A is a BL-algebra of global sections, U is an open subset of X and σ is a section over U, we say that σ is *locally in the BL-algebra of global sections A* if

(*) there are an open covering $(U_i)_{i \in I}$ of U and a family $(\sigma_i)_{i \in I}$ of elements of A such that $\sigma|_{U_i} = \sigma_i|_{U_i}$ for all $i \in I$.

The following lemma follows immediately from Proposition 3.1(iv).

Lemma 3.2. *Let (F, p, X) be a BL-sheaf space and A a BL-algebra of global sections such that every section over an open subset of X is locally in the BL-algebra A. Then the family $\{\sigma(U) \mid U$ is open in X, $\sigma \in A\}$ is a basis for the topology of F.*

Proposition 3.3. *Let (F, p, X) be a BL-sheaf space and A a BL-algebra of global sections. The following are equivalent:*
(i) every section over an open subset of X is locally in the BL-algebra A;
(ii) for each $x \in X$, the BL-morphism p_x^A is onto.

Proof. (i)$\Rightarrow$(ii) Let $x \in X$ and $a \in F_x$, that is $a \in F$ such that $p(a) = x$. Applying Proposition 3.1(ii), there is an open neighborhood U of x and a section σ over U such that $\sigma(x) = a$. By (i), we get an open covering $(U_i)_{i \in I}$ of U and a family $(\sigma_i)_{i \in I}$ of sections from A such that $\sigma|_{U_i} = \sigma_i|_{U_i}$ for all $i \in I$. Since $x \in U$, we have that $x \in U_k$ for some $k \in I$. It follows that $\sigma_k(x) = \sigma(x) = a$. Hence, we have got $\sigma_k \in A$ such that $p_x^A(\sigma_k) = a$. That is, p_x^A is onto.
(ii)$\Rightarrow$(i) Let U be an open subset of X and σ a section over U. For each $x \in U$, we have that $\sigma(x) \in F_x$, hence, by (ii), there is $\tau^x \in A$ such that $\tau^x(x) = \sigma(x)$. Applying Proposition 3.1(iii) and (i), it follows that $\tau^x|_U \in \Gamma(U, F)$ and the subset $U_x = [\tau^x|_U = \sigma]$ is an open subset of U such that $x \in U_x$. Thus, we have got an open covering $(U_x)_{x \in U}$ of U and a family $(\tau^x)_{x \in U}$ of sections from A such that $\tau^x|_{U_x} = (\tau^x|_U)|_{U_x} = \sigma|_{U_x}$ for all $x \in U$. □ □

Let (F, p, X) be a BL-sheaf space and $\sigma \in \Gamma(Y, F)$ a section over $Y \subseteq X$. The *cosupport* of σ, *cosupp*(σ), is the closed hull in the subspace Y of the set of those points $x \in Y$ for which $\sigma(x) \neq 1_x$:

cosupp$(\sigma) = \overline{\{x \in Y \mid \sigma(x) \neq 1_x\}}$.

It is easy to see that $(\textit{cosupp}(\sigma))^c = [\sigma = \underline{1}|_Y]$.

Let X and Y be topological spaces and $f : Y \to X$ a continuous function. Let (F, p, X) and (G, q, Y) be two BL-sheaf spaces. A *morphism $\alpha : F \to G$ over f* is a family $(\alpha_y : F_{f(y)} \to G_y)_{y \in Y}$ of BL-morphisms satisfying the following condition:

If U is open in X and $\sigma \in \Gamma(U, F)$, define $\beta : f^{-1}(U) \to G$ by

$\beta(y) = \alpha_y(\sigma(f(y)))$.

Then β is continuous, and therefore $\beta \in \Gamma(f^{-1}(U), G)$.

We shall write $\beta = \alpha_{\#}^{U}(\sigma)$.

It follows that a morphism $\alpha : F \to G$ over f induces a BL-morphism $\alpha_{\#}^{U} : \Gamma(U, F) \to \Gamma(f^{-1}(U), G)$ for all open U in X. We shall denote $\alpha_{\#}^{X}$ by $\alpha_{\#}$. Since $f^{-1}(X) = Y$, $\alpha_{\#}$ is a BL-morphism between the BL-algebras of global sections $\Gamma(X, F)$ and $\Gamma(Y, G)$.
An example of a morphism over f is given by the canonical mapping from a BL-sheaf space (F, p, X) to the BL-sheaf space $(f^{-1}(F), q, Y)$, *induced* by f and (F, p, X), defined as follows.
Define $f^{-1}(F) = \{(y, a) \in Y \times F \mid f(y) = p(a)\} = \bigcup_{y \in Y}\{y\} \times F_{f(y)}$ and $q : f^{-1}(F) \to Y$ by $q(y, a) = y$. Then for all $y \in Y$, $f^{-1}(F)_y = \{y\} \times F_{f(y)}$. For each $y \in Y$, define $i_y : F_{f(y)} \to f^{-1}(F)_y$ by $i_y(a) = (y, a)$. We get easily that i_y is a bijection. We make $f^{-1}(F)_y$ a BL-algebra by transporting the BL-structure of $F_{f(y)}$ to $f^{-1}(F)_y$ by means of i_y.
Thus, we have got a BL-sheaf space $(f^{-1}(F), q, Y)$ and a morphism $i : F \to f^{-1}(F)$ over f, where i is the family $(i_y)_{y \in Y}$.

A *morphism of BL-sheaf spaces* $(f, \alpha) : (F, p, X) \to (G, q, Y)$ consists of a continuous function $f : Y \to X$ and a morphism $\alpha : F \to G$ over f.
An *isomorphism of BL-sheaf spaces* is a morphism (f, α) such that f is a homeomorphism and α_y is an isomorphism of BL-algebras for all $y \in Y$.
If $(f, \alpha) : (F, p, X) \to (G, q, Y)$ and $(g, \beta) : (G, q, Y) \to (H, r, Z)$ are two morphisms of BL-sheaf spaces, then their composition is the morphism $(f \circ g, \beta \circ \alpha)$, where $(\beta \circ \alpha)_z = \beta_z \circ \alpha_{g(z)}$ for all $z \in Z$.

Let (F, p, X) and (G, q, X) be BL-sheaf spaces over the same topological space X. If $(\alpha_x : F_x \to G_x)_{x \in X}$ is a family of functions, then we can define a function $\alpha : F \to G$ by $\alpha(a) = \alpha_x(a)$, where $x \in X$ is unique such that $a \in F_x$. Conversely, a function $\alpha : F \to G$ can be seen as a family $(\alpha_x : F_x \to G_x)_{x \in X}$, where $\alpha_x = \alpha \mid F_x$ for all $x \in X$.

Proposition 3.4. $(1_X, \alpha) : (F, p, X) \to (G, q, X)$ *is a morphism of BL-sheaf spaces iff* $\alpha : F \to G$ *is a continuous function such that* $q \circ \alpha = p$ *and* $\alpha_x : F_x \to G_x$ *is a BL-morphism for all* $x \in X$.

We shall denote by BL the category of nontrivial BL-algebras and BL-morphisms and by $BL - ShSp$ the category of BL-sheaf spaces and morphisms of BL-sheaf spaces.
Define $\mathcal{S}(F, p, X) = \Gamma(X, F)$ for any BL-sheaf space (F, p, X) and $\mathcal{S}(f, \alpha) = \alpha_{\#}$ for every morphism $(f, \alpha) : (F, p, X) \to (G, q, Y)$. Then

Proposition 3.5. $\mathcal{S} : BL - ShSp \to BL$ *is a functor, called the* section *functor.*

4 Compact BL-sheaf spaces

Throughout, BL-algebras are nontrivial and X will be assumed to denote a Hausdorff topological space.

A BL-sheaf space (F, p, X) is called *completely regular* if it satisfies the following:

(CR) for each $x \in X$ and closed set $C \subseteq X$ not containing x, there is $\sigma \in \Gamma(X, F)$ such that $\sigma(x) = 0_x$ and $\sigma|_C = \underline{1}|C$.

A completely regular BL-sheaf space (F, p, X) is called *compact* if the topological space X is compact.

The following lemma gives equivalent characterizations of completely regular BL-sheaf spaces.

Lemma 4.1. *Let (F, p, X) be a BL-sheaf space. The following are equivalent:*
(i) (F, p, X) is completely regular;
(ii) for each $x \in X$ and every open neighborhood U of x there is $\sigma \in \Gamma(X, F)$ such that $\sigma(x) = 0_x$ and $\sigma(y) = 1_y$ for all $y \notin U$;
(iii) for each $x \in X$ and every open neighborhood U of x there is $\sigma \in \Gamma(X, F)$ such that $\sigma(x) = 0_x$ and $cosupp(\sigma) \subseteq U$.

Proof. (i)⇒(ii) Let $C = U^c$. Then C is a closed subset of X such that $x \notin C$, and applying (i) we get (ii).
(ii)⇒(i) Take $U = C^c$ and apply (ii).
(ii)⇔(iii) Apply the fact that $(cosupp(\sigma))^c = [\sigma = \underline{1}]$. □ □

Proposition 4.2. *Let (F, p, X) be a completely regular BL-sheaf space. Then*
(i) X is a regular topological space;
(ii) every section over an open subset of X is locally in the BL-algebra $\Gamma(X, F)$ of global sections of the BL-sheaf space;
(iii) the family $[\sigma = \underline{1}]_{\sigma \in \Gamma(X,F)}$ form a basis for the topology of X;
(iv) $F_x \cong A/Ker(p_x)$ for all $x \in X$.

Proof. (i) Let $x \in X$ and U be an open neighborhood of x. Applying Lemma 4.1(iii), there is $\sigma \in \Gamma(X, F)$ such that $\sigma(x) = 0_x$ and $cosupp(\sigma) \subseteq U$. Hence, $x \in [\sigma = \underline{0}]$ and, since F_y is nontrivial for all $y \in X$, we have that $0_y \neq 1_y$ for all $y \in X$, so $x \in [\sigma = \underline{0}] \subseteq cosupp(\sigma)$. Hence, there is a closed neighborhood $C = cosupp(\sigma)$ of x such that $C \subseteq U$. Thus, the closed neighborhoods of x form a basis for neighborhoods, so X is regular.
(ii) We shall prove that (ii) from Proposition 3.3 is satisfied with $A = \Gamma(X, F)$. Hence, we have to show that for each $x \in X$, p_x is onto. Let $a \in F_x$, that is $a \in F$ such that $p(a) = x$. Applying Proposition 3.1(ii), there is an open neighborhood U of x and a section τ over U such that $\tau(x) = a$. By Lemma 4.1(iii), there is $\theta \in \Gamma(X, F)$ such that $\theta(x) = 0_x$ and $cosupp(\theta) \subseteq U$. Let $\sigma : X \to F$ defined by $\sigma(y) = \theta(y)^- \to_y \tau(y)$ for $y \in U$ and $\sigma(y) = 1_y$ for $y \notin U$. It is obvious that $p \circ \sigma = 1_X$ and that $p_x(\sigma) = \sigma(x) = \theta(x)^- \to_x \tau(x) = 0_x^- \to_x a = 1_x \to a = a$. It remains to prove that σ is continuous. Since $cosupp(\theta) \subseteq U$, we get that $U \cup (cosupp(\theta))^c = X$. Let us prove that $\sigma(y) = 1_y$ for all $y \in (cosupp(\theta))^c$. If $y \notin U$, then $\sigma(y) = 1_y$ by the definition of σ. If $y \in U \cap (cosupp(\theta))^c$, then $\theta(y) = 1_y$ and $\sigma(y) = \theta(y)^- \to_y \tau(y) = 1_y^- \to_y \tau(y) = 0_y \to \tau(y) = 1_y$. Hence, we have got that $\sigma|_U, \sigma|_{(cosupp(\theta))^c}$ are continuous and $U, (cosupp(\theta))^c$ form an open covering of X. It follows that σ is continuous. Thus, we have obtained $\sigma \in \Gamma(X, F)$ such that $p_x(\sigma) = a$.

(iii) We have that $[\sigma = \underline{1}]$ is open in X for all $\sigma \in \Gamma(X, F)$. We shall prove that for any $x \in X$ and any open neighborhood U of x there is $\sigma \in \Gamma(X, F)$ such that $x \in [\sigma = \underline{1}] \subseteq U$. From this we get immediately that $[\sigma = \underline{1}]_{\sigma \in \Gamma(X,F)}$ form a basis for the topology of X. Applying Lemma 4.1(iii), there is $\tau \in \Gamma(X, F)$ such that $\tau(x) = 0_x$ and $cosupp(\tau) \subseteq U$. Let $\sigma = \tau^-$. Then, $\sigma(x) = (\tau(x))^- = 0_x^- = 1_x$, hence $x \in [\sigma = \underline{1}]$. If $y \in [\sigma = \underline{1}]$, then $\sigma(y) = 1_y$, that is $(\tau(y))^- = 1_y$. It follows that $\tau(y) \neq 1_y$, since $0_y \neq 1_y$, so $y \in cosupp(\tau) \subseteq U$. Hence, we have proved that $[\sigma = \underline{1}]$ is an open neighborhood of x contained in U.
(iv) We have proved at (ii) that the BL-morphism $p_x : A \to F_x$, $p_x(\sigma) = \sigma(x)$ is onto. Hence, $F_x \cong A/Ker(p_x)$. □ □

Let A be a BL-algebra of global sections of the BL-sheaf space (F, p, X). We say that A is *completely regular in the BL-sheaf space* (F, p, X) if for each $x \in X$ and closed set $C \subseteq X$ not containing x, there is $\sigma \in A$ such that $\sigma(x) = 0_x$ and $\sigma|_C = \underline{1}|_C$.
If A is completely regular in (F, p, X) and X is compact, then A is said to be *compact in the BL-sheaf space* (F, p, X).
It is easy to see that, as in Lemma 4.1, A is completely regular in the BL-sheaf space (F, p, X) iff for each $x \in X$ and every open neighborhood U of x there is $\sigma \in A$ such that $\sigma(x) = 0_x$ and $\sigma(y) = 1_y$ for all $y \notin U$. The following result extends Proposition 4.2(i) and (iii) and its proof is similar.

Lemma 4.3. *Let A be a BL-algebra of global sections that is completely regular in (F, p, X). Then*
(i) X is regular;
(ii) the family $[\sigma = \underline{1}]_{\sigma \in A}$ form a basis for the topology of X.

The following lemma collects some obvious facts that will be used in the sequel.

Lemma 4.4. *Let (F, p, X) be a BL-sheaf space.*
(i) (F, p, X) is completely regular (compact) iff the BL-algebra $\Gamma(X, F)$ of global sections is completely regular (compact) in (F, p, X);
(ii) Suppose that A and B are BL-algebras of global sections such that $A \subseteq B$. If A is completely regular (compact) in (F, p, X), then B is completely regular (compact) in (F, p, X);
(iii) If there is a BL-algebra A of global sections that is completely regular (compact) in (F, p, X), then (F, p, X) is completely regular (compact).

Proposition 4.5. *Let A be a BL-algebra of global sections that is compact in (F, p, X) and suppose that every global section is locally in A. Then A is necessarily the BL-algebra $\Gamma(X, F)$.*

Proof. Let $\sigma \in \Gamma(X, F)$. Since σ is locally in A, it follows that there are an open covering $(U_i)_{i \in I}$ of X and a family $(\sigma_i)_{i \in I}$ of elements of A such that $\sigma|_{U_i} = \sigma_i|_{U_i}$ for all $i \in I$. For each $x \in X$, there is $i_x \in I$ such that $x \in U_{i_x}$ and applying the fact that A is completely regular in (F, p, X), we get $\tau_{i_x} \in A$ such that $\tau_{i_x}(x) = 0_x$ and $\tau_{i_x}(y) = 1_y$ for all $y \notin U_{i_x}$. Let us denote $U_{i_x} \cap [\tau_{i_x} = \underline{0}]$

by V_x. Then, $x \in V_x \subseteq U_{i_x}$ for all $x \in X$, so the family $(V_x)_{x\in X}$ is an open covering of X. Since X is compact, it follows that there are $x_1, \cdots, x_n \in X$ such that $X = V_{x_1} \cup \cdots \cup V_{x_n}$. Let us denote V_{x_k} by V_k, i_{x_k} by i_k and $\tau_{i_{x_k}}$ by τ_k for all $k = \overline{1,n}$. We shall prove that $\sigma = \bigwedge_{k=\overline{1,n}}(\sigma_{i_k} \vee \tau_k)$. Let $x \in X$ and $J = \{k = \overline{1,n} \mid x \in U_{i_k}\}$. It is obvious that J is nonempty, since $\bigcup_{k=\overline{1,n}} U_{i_k} = X$. We have that $\sigma_{i_k}(x) = \sigma(x)$ for all $k \in J$ and $x \notin U_{i_k}$ for all $k \notin J$, so $\tau_k(x) = 1_x$ for all $k \notin J$. It follows that $[\bigwedge_{k\in J}(\sigma_{i_k}(x) \vee \tau_k(x))] \wedge [\bigwedge_{k\notin J}(\sigma_{i_k}(x) \vee \tau_k(x))] = [\bigwedge_{k\in J}(\sigma(x) \vee \tau_k(x))] \wedge [\bigwedge_{k\notin J}(\sigma_{i_k}(x) \vee 1_x)] = \sigma(x) \vee \bigwedge_{k\in J} \tau_k(x)$. Since $X = \bigcup_{k=\overline{1,n}} V_k$, there is $j = \overline{1,n}$ such that $x \in V_j$, so $\tau_j(x) = 0_x$ and $j \in J$, since $V_j \subseteq U_{i_j}$. It follows that $(\bigwedge_{k\in J} \tau_k)(x) = 0_x$, hence $[\bigwedge_{k=\overline{1,n}}(\sigma_{i_k} \vee \tau_k)](x) = \sigma(x)$. Thus, $\sigma = \bigwedge_{k=\overline{1,n}}(\sigma_{i_k} \vee \tau_k)$, hence $\sigma \in A$.
□ □

4.1 The compactness theorem

In the sequel, A will be a BL-algebra of global sections of the BL-sheaf space (F, p, X).
For each $x \in X$, let us denote $K_x = Ker(p_x^A) = \{\sigma \in A \mid \sigma(x) = 1_x\}$. Since A is nontrivial, it follows that K_x is a proper filter of A.
A filter T of A is called *fixed* if there is $x \in X$ such that $T \vee K_x$ is a proper filter of A. Otherwise, T is said to be a *free* filter of A.

Lemma 4.6. *Let A be a BL-algebra of global sections of (F, p, X), P a prime filter and M a maximal filter of A. Then*
(i) M is fixed iff M contains the filter K_x for some $x \in X$;
(ii) if M_P is the unique maximal filter that contains P, then P is fixed iff M_P is fixed;
(iii) if P contains the filter K_x for some $x \in X$, then P is fixed.

Proof. (i) Suppose that M is fixed, so there is $x \in X$ such that $M \vee K_x$ is a proper filter of A. Since $M \subseteq M \vee K_x$ and M is maximal, it follows that $M \vee K_x = M$, hence $K_x \subseteq M$. Conversely, if $K_x \subseteq M$ for some $x \in X$, we get that $M \vee K_x = M$, so $M \vee K_x$ is a proper filter of A. That is, M is fixed.
(ii) If M_P is fixed, then, by (i), there is $x \in X$ such that $K_x \subseteq M_P$. Since $P \subseteq M_P$, we have that $P \vee K_x \subseteq M_P$, hence $P \vee K_x$ is a proper filter of A, i.e. P is fixed. Conversely, suppose that P is fixed, that is $P \vee K_x$ is proper for some $x \in X$. We get that M_P and $P \vee K_x$ are proper filters containing the prime filter P, so applying Proposition 2.2 and the fact that M_P is maximal, it follows that $P \vee K_x \subseteq M_P$. Hence, $K_x \subseteq M_P$, so by (i), M_P is fixed.
(iii) Since $K_x \subseteq M_P$, we get that M_P is fixed, by (i). Applying (ii), we obtain that P is also fixed. □ □

Lemma 4.7. *Let A be a BL-algebra of global sections of (F, p, X). The following are equivalent*
(i) every proper filter of A is fixed;
(ii) every prime filter of A is fixed;
(iii) every maximal filter of A is fixed.

Proof. (i)$\Rightarrow$ (ii) Obviously.
(ii)$\Rightarrow$ (iii) Apply the fact that $Max(A) \subseteq Spec(A)$, by Proposition 2.4.
(iii)$\Rightarrow$(i) Let F be a proper filter of A. By Proposition 2.1, there is a maximal filter M such that $F \subseteq M$. Since M is fixed, we get $x \in X$ such that $K_x \subseteq M$. We have that $F, K_x \subseteq M$, so $F \vee K_x \subseteq M$. Hence, $F \vee K_x$ is a proper filter of A, that is F is fixed. □ □

Lemma 4.8. *Let A be a BL-algebra of global sections of (F, p, X) and suppose that X is compact. Then*
(i) for every prime filter P of A there is $x \in X$ such that $K_x \subseteq P$;
(ii) every proper filter of A is fixed.

Proof. (i) Let P be a prime filter of A and suppose that $K_x \not\subseteq P$ for any $x \in X$. That is for any $x \in X$ there is $\sigma^x \in K_x$ such that $\sigma^x \notin P$. Since $\sigma^x \in K_x$, we get that $\sigma^x(x) = 1_x$, that is $x \in [\sigma^x = \underline{1}]$. Thus, $X = \bigcup_{x\in X}[\sigma^x = \underline{1}]$, hence the family $[\sigma^x = \underline{1}]_{x\in X}$ is an open covering of X. Since X is compact, it follows that there are $x_1, \cdots, x_n \in X$ such that $X = \bigcup_{i=1}^{n}[\sigma_i = \underline{1}]$, where σ_i denotes σ^{x_i} for $i = \overline{1,n}$. It follows immediately that $\sigma_1 \vee \cdots \vee \sigma_n = \underline{1} \in P$. Since P is prime, we obtain that $\sigma_i \in P$ for some $i = \overline{1,n}$. Thus, we have got a contradiction.
(ii) Applying (i) and Lemma 4.6(iii), we obtain that every prime filter of A is fixed. Now apply Lemma 4.7 to get that every proper filter of A is fixed. □ □

In the following, we shall denote by $Spec_X(A)$ the set of prime filters of A that are fixed and by $Max_X(A)$ the set of maximal filters of A that are fixed.

Lemma 4.9. *Suppose that A is completely regular in (F, p, X). Then*
(i) for any $P \in Spec_X(A)$ there is a unique $x \in X$ such that $K_x \subseteq M_P$, where M_P is the unique maximal filter that contains P;
(ii) for any $M \in Max_X(A)$ there is a unique $x \in X$ such that $K_x \subseteq M$.

Proof. (i) The existence of $x \in X$ such that $K_x \subseteq M_P$ follows from Lemma 4.6. It remains to prove the unicity. Let us suppose that there is $y \neq x$ such that $K_y \subseteq M_P$. Since X is Hausdorff, there is an open neighborhood U of x such that $y \notin U$. Applying now Lemma 4.1(ii), there is $\sigma \in A$ such that $\sigma(x) = 0_x$ and $\sigma(z) = 1_z$ for all $z \notin U$. It follows that $\sigma(y) = 1_y$, so $\sigma \in K_y \subseteq M_P$ and $\sigma^-(x) = 1_x$, hence $\sigma^- \in K_x \subseteq M_P$. We have got that $\sigma, \sigma^- \in M_P$, hence $\sigma \odot \sigma^- = \underline{0} \in M_P$. Thus, we have obtained that M_P is not proper, that is a contradiction.
(ii) By (i). □ □

If A is completely regular in (F, p, X), then, by the above lemma, we can define a function $\mathbf{s} : Spec_X(A) \to X$ that assigns to each $P \in Spec_X(A)$ the unique $x \in X$ such that $K_x \subseteq M_P$. We shall denote by $\mathbf{m}$ its restriction to $Max_X(A)$. Then $\mathbf{m}$ assigns to every fixed maximal filter M of A the unique $x \in X$ such that $K_x \subseteq M$.

Corollary 4.10. *Let A be a BL-algebra of global sections of (F, p, X) and suppose that X is compact. Then for every prime filter P of A there is a unique $x \in X$ such that $K_x \subseteq P$.*

Proof. Apply Lemmas 4.8 and 4.9. □ □

Lemma 4.11. *Suppose that A is completely regular in (F,p,X). Then for any $M \in Max_X(A)$, $K_{\mathbf{m}(M)} \subseteq O(M)$.*

Proof. Let $x = \mathbf{m}(M)$ and $\sigma \in K_x$. We get that $\sigma(x) = 1_x$, so $x \in [\sigma = \underline{1}]$. Applying the fact that A is completely regular in (F,p,X), we get $\tau \in A$ such that $\tau(x) = 0_x$ and $\tau(y) = 1_y$ for all $y \notin [\sigma = \underline{1}]$. It is clear that $\sigma \vee \tau = \underline{1}$. From $\tau(x) = 0_x$, it follows that $\tau^-(x) = 1_x$, so $\tau^- \in K_x \subseteq M$. Since M is proper, we must have $\tau \notin M$. Hence, there is $\tau \notin M$ such that $\sigma \vee \tau = \underline{1}$, that is $\sigma \in O(M)$. □ □

Lemma 4.12. *Let (F,p,X) be a completely regular local BL-sheaf space and $A = \Gamma(X,F)$. Then*
(i) for any $x \in X$ there is a unique $M \in Max(A)$ such that $K_x \subseteq M$;
(ii) $K_{\mathbf{m}(M)} = O(M)$ for any $M \in Max_X(A)$.

Proof. (i) By Proposition 4.2(v) and the fact that $Ker(p_x) = K_x$, it follows that $F_x \cong A/K_x$ for all $x \in X$. Hence, A/K_x is local for any $x \in X$. Apply now Proposition 2.14.
(ii) Applying Proposition 4.11, we have that $K_{\mathbf{m}(M)} \subseteq O(M)$. Let us prove the converse inclusion. If we denote $x = \mathbf{m}(M)$, then $K_x \subseteq M$. Let $\sigma \in O(M)$, so there is $\tau \notin M$ such that $\sigma \vee \tau = \underline{1}$. Since F_x is local, its unique maximal filter is $N_x = \{a \in F_x \mid ord(a) = \infty\}$. By Proposition 2.10, we have that $p_x^{-1}(N_x)$ is a maximal filter of A and it is easy to see that $K_x \subseteq p_x^{-1}(N_x)$. Since $K_x \subseteq p_x^{-1}(N_x)$, $K_x \subseteq M$ and $p_x^{-1}(N_x), M$ are maximal filters of A, applying (i) it follows that $p_x^{-1}(N_x) = M$. Now, $\tau \notin M$ implies $\tau \notin p_x^{-1}(N_x)$, so $ord(\tau(x))) < \infty$. Thus, there is $n \in \omega - \{0\}$ such that $(\tau(x))^n = 0_x$. Since $\sigma \vee \tau = \underline{1}$, we get that $\sigma(x) \vee_x \tau(x) = 1_x$, so $(\sigma(x))^n \vee_x (\tau(x))^n = 1_x$, that is $(\sigma(x))^n = 1_x$, hence $\sigma(x) = 1_x$. Thus, we have got that $\sigma \in K_x$. □ □

Proposition 4.13. *Let A be completely regular in (F,p,X). Then $\mathbf{s}$ is onto and $\mathbf{m}$ is continuous and onto.*

Proof. Let $x \in X$. Then K_x is a proper filter of A, so, by Proposition 2.1, there is a maximal filter M such that $K_x \subseteq M$. Applying Lemma 4.6(i), we get that M is fixed. Hence, $M \in Max_X(A)$ is such that $\mathbf{m}(M) = x$. Thus, $\mathbf{m}$ is onto and, obviously, $\mathbf{s}$ is also onto. Let us prove now that $\mathbf{m}$ is continuous. Let $M \in Max_X(A)$, $x = \mathbf{m}(M)$ and U an open neighborhood of x. Since A is completely regular in (F,p,X)), there is $\sigma \in A$ such that $\sigma(x) = 0_x$ and $\sigma(y) = 1_y$ for all $y \notin U$. Let $V = d(\sigma) \cap Max_X(A) = \{N \in Max_X(A) \mid \sigma \notin N\}$. Then V is an open subset of $Max_X(A)$. Since $\sigma(x) = 0_x$, we get that $\sigma^-(x) = 1_x$, that is $\sigma^- \in K_x \subseteq M$. It follows that $\sigma \notin M$, hence $M \in V$. Let us prove that $\mathbf{m}(V) \subseteq U$. Let $N \in V$ and $y = \mathbf{m}(N)$, so $K_y \subseteq N$. If $y \notin U$, then $\sigma(y) = 1_y$, so $\sigma \in K_y$, hence $\sigma \in N$. This contradicts the fact that $N \in d(\sigma)$. It follows that $y \in U$. Thus, we have proved that V is an open neighborhood of M such that $\mathbf{m}(V) \subseteq U$. That is, $\mathbf{m}$ is continuous at M. □ □

Suppose that A is compact in (F, p, X). Then, by Lemma 4.8, we have that $Spec_X(A) = Spec(A)$ and, by Corollary 4.10, $\mathbf{s} : Spec(A) \to X$ assigns to every prime filter P of A the unique $x \in X$ such that $K_x \subseteq P$. We obtain the following corollary.

Corollary 4.14. *Let A be compact in the BL-sheaf space (F, p, X). Then $\mathbf{s}$ and $\mathbf{m}$ are continuous, closed and onto.*

Proof. We get that $\mathbf{s}$ is continuous in a similar manner with the proof of continuity of $\mathbf{m}$ from Proposition 4.13 . To obtain that the functions are closed, apply [20, Theorem 7.2.2, p. 71], since $\mathbf{s}, \mathbf{m}$ are continuous and onto, $Max(A)$ and $Spec(A)$ are compact and X is Hausdorff. □ □

Theorem 4.15. (The compactness theorem)
Suppose that A is completely regular in the BL-sheaf space (F, p, X). The following are equivalent
(i) the topological space X is compact;
(ii) every proper filter of A is fixed;
(iii) every maximal filter of A is fixed;
(iv) every prime filter of A is fixed;
(v) A is compact in the BL-sheaf space (F, p, X).

Proof. (i)⇔(v) By definition.
(ii)⇔(iii)⇔(iv) By Lemma 4.7.
(i)⇒(ii) Apply Lemma 4.8.
(ii)⇒(i) We have that $Max_X(A) = Max(A)$ and $\mathbf{m} : Max(A) \to X$. Since $\mathbf{m}$ is continuous and onto and $Max(A)$ is compact, applying a known result of topology, it follows that X is also compact (see, e.g., [20, Theorem 7.2.1, p.71]). □ □

Proposition 4.16. *If (F, p, X) is a compact BL-sheaf space and $A = \Gamma(X, F)$, then*
$\mathbf{m}$ *is a homeomorphism iff (F, p, X) is a local BL-sheaf space.*

Proof. Applying Propositions 2.14 and 4.2(iv), it follows that $\mathbf{m}$ is injective iff for any $x \in X$ there is a unique maximal filter M of A such that $\mathbf{m}(M) = x$ iff for any $x \in X$ there is a unique maximal filter M of A such that $K_x \subseteq M$ iff for all $x \in X$, A/K_x is local iff for all $x \in X$, F_x is a local BL-algebra. Hence, if $\mathbf{m}$ is a homeomorphism, then (F, p, X) is a local BL-sheaf space. Conversely, if (F, p, X) is local, then $\mathbf{m}$ is injective. We have that $\mathbf{m}$ is bijective, continuous and closed, by Corollary 4.14. Hence, $\mathbf{m}$ is a homeomorphism. □ □

Let (F, p, X) be a compact local BL-sheaf space and $A = \Gamma(X, F)$. By the proof of the above proposition, we can define a function $\mathbf{n} : X \to Max(A)$, that associates with every $x \in X$ the unique maximal filter M of A such that $K_x \subseteq M$. It is easy to see that

Proposition 4.17. *Let (F, p, X) be a compact local BL-sheaf space. Then $\mathbf{n}$ is the inverse of $\mathbf{m}$, hence $\mathbf{n} : X \to Max(A)$ is also a homeomorphism.*

5 Compact representations of BL-algebras

By a *sheaf representation* (or simply *representation*) of a non-trivial BL-algebra A will be meant a BL-morphism
$\varphi : A \to \Gamma(X, F)$
from A to the BL-algebra of global sections of a BL-sheaf space (F, p, X).
Hence, $\varphi(A)$ is a BL-algebra of global sections of (F, p, X). In a representation φ, each $a \in A$ determines a global section $\varphi(a)$; in particular, for every $x \in X$, $\varphi(a)(x)$ is an element of the stalk F_x.
For each $x \in X$, we define
$\varphi_x : A \to F_x, \quad \varphi_x(a) = \varphi(a)(x)$ for all $a \in A$,
$K_x = Ker(\varphi_x) = \{a \in A \mid \varphi(a)(x) = 1_x\}$.
Since $\varphi_x = p_x \circ \varphi$, we have that φ_x is a BL-morphism, so K_x is a proper filter of A for every $x \in X$.
It is easy to see that $Ker(\varphi) = \bigcap_{x\in X} K_x$, hence φ is a monomorphism iff $\bigcap_{x\in X} K_x = \{1\}$.
For every $a \in A$, we shall use the following notation:
$V(a) = [\varphi(a) = \underline{1}] = \{x \in X \mid \varphi(a)(x) = 1_x\} = \{x \in X \mid a \in K_x\}$.
By Proposition 3.1(i), $V(a)$ is open in X for all $a \in A$.

Let us remind that a *representation* of a non-trivial BL-algebra A as a *subdirect product* of non-trivial BL-algebras $(A_i)_{i\in I}$ (or a *subdirect representation* of A) consists of a monomorphism
$\alpha : A \to \prod_{i\in I} A_i$
such that for all $j \in I$ the BL-morphism
$A \xrightarrow{\alpha} \prod_{i\in I} A_i \xrightarrow{\pi_j} A_j$
is surjective.

Proposition 5.1. *Any sheaf representation $\varphi : A \to \Gamma(X, F)$ such that φ is a monomorphism determines a subdirect representation of A*

Proof. Since φ is a monomorphism, we have that $\bigcap_{x\in X} K_x = \{1\}$. Applying now a general result of universal algebra (see, e.g., [2], Lemma II.8.2, p. 57) it follows that $\alpha : A \to \prod_{x\in X} A/K_x$, defined by $\alpha(a)(x) = a/K_x$, is a subdirect representation of A. □ □

A *filter space* of a BL-algebra A is a family $(T_x)_{x\in X}$ of proper filters of A, indexed by a topological space X.
Let $\varphi : A \to \Gamma(X, F)$ be a representation of A. The filter space $(K_x)_{x\in X}$ will be called the *representation space* of the representation, and the filters indexed the *representation filters*. The topology generated by the family $(V(a))_{a\in A}$ of subsets of X is called the *representation topology* on the space X. Then, any topology on X contains the representation topology.
We say that a filter space $(T_x)_{x\in X}$ *canonically determines* a representation of A if there is a representation $\varphi : A \to \Gamma(X, F)$ such that $T_x = K_x$ for all $x \in X$.

In the sequel, we shall give an existence theorem for representations of BL-algebras.

Let A be a nontrivial BL-algebra and $(T_x)_{x\in X}$ a filter space of A such that the subset $V(a) = \{x \in X \mid a \in T_x\}$ is open in X for all $a \in A$. Then a BL-sheaf space (F_A, p_A, X) and a representation $\varphi : A \to \Gamma(X, F_A)$ can be constructed in the following way, given in [5] for universal algebra. Let F_A be the disjoint union of the sets $(A/T_x)_{x\in X}$ and $p_A : F_A \to X$ the canonical projection, so $p_A^{-1}(\{x\}) = A/T_x$ for all $x \in X$. For all $x \in X$, T_x is a proper filter of A, so A/T_x is a nontrivial BL-algebra. For each $a \in A$, define the map $[a] : X \to F_A$ by $[a](x) = a/T_x$. Endow F_A with the topology generated by the family $\{[a](U) \mid a \in A \text{ and } U \text{ is open in } X\}$. Applying [5, Corollary 2], we get that (F_A, p_A, X) is a sheaf space of BL-algebras and the function $\varphi : A \to \Gamma(X, F_A)$, defined by $\varphi(a) = [a]$ for all $a \in A$, is a representation of A. It is easy to see that $K_x = T_x$ for all $x \in X$.
Hence, we get the following theorem:

Theorem 5.2. *Let A be a nontrivial BL-algebra and $(T_x)_{x\in X}$ a filter space of A such that the subset $V(a) = \{x \in X \mid a \in T_x\}$ is open in X for all $a \in A$. Then $(T_x)_{x\in X}$ canonically determines a representation of A.*

Corollary 5.3. *Any subdirect representation of A determines a sheaf representation of A.*

Proof. Let $\alpha : A \to \prod_{x\in X} A_x$ be a subdirect representation of A. For any $x \in X$, take $T_x = Ker(\pi_x \circ \alpha)$ and consider on X the topology generated by the family $(V(a))_{a\in A}$. Apply now Theorem 5.2. □ □

From now on, X will be assumed to denote a Hausdorff topological space. We shall define completely regular and compact representations and, finally, we shall prove that any compact representation arises canonically from a filter space of the BL-algebra satisfying certain conditions.

Thus, a representation $\varphi : A \to \Gamma(X, F)$ of a BL-algebra A in a BL-sheaf space (F, p, X) will be said to be a *completely regular representation of* A if φ is a monomorphism and $\varphi(A)$ is completely regular in (F, p, X). A *compact representation* of A is a monomorphism $\varphi : A \to \Gamma(X, F)$ such that $\varphi(A)$ is compact in (F, p, X). Hence, a compact representation is a completely regular representation $\varphi : A \to \Gamma(X, F)$ with the property that X is compact.
By Proposition 5.1, we get that any completely regular (compact) representation determines a subdirect representation of A.

Proposition 5.4. *Let $\varphi : A \to \Gamma(X, F)$ be a completely regular representation of A. Then*
(i) for any distinct $x, y \in X$, there is $a \in A$ such that $\varphi(a)(x) = 0_x$ and $\varphi(a)(y) = 1_y$;
(ii) the topology on X is the representation topology.

Proof. (i) Since X is Hausdorff, we have that $\{y\}$ is closed in X. Apply now the fact that $\varphi(A)$ is completely regular in (F, p, X) for the closed set $\{y\}$ and $x \notin \{y\}$.
(ii) As we have noticed, the topology on X contains the representation topology. For the converse, apply Proposition 4.3(ii). □ □

For any BL-algebra A, a family $(T_x)_{x\in X}$ of proper filters of A will be said to be *coprime* if $\bigcap_{x\in X} T_x = \{1\}$ and for any distinct $x, y \in X$ we have

$T_x \vee T_y = A$.

The family $(T_x)_{x\in X}$ is called *strongly coprime* if $\bigcap_{x\in X} T_x = \{1\}$ and for any $x \in X$ and $a \in T_x$, we have

$T_x \vee \bigcap\{T_y \mid y \in X \text{ and } a \notin T_y\} = A$.

In the sequel, let us consider a filter space $(T_x)_{x\in X}$ of A such that the subset $V(a) = \{x \in X \mid a \in T_x\}$ is open in X for all $a \in A$. By Theorem 5.2, there is a representation $\varphi : A \to \Gamma(X, F)$ of A such that $T_x = K_x = \{a \in A \mid \varphi(a)(x) = 1_x\}$ for all $x \in X$.

Lemma 5.5. *If φ is completely regular, then the family $(T_x)_{x\in X}$ is coprime.*

Proof. Since φ is a monomorphism, we have that $\bigcap_{x\in X} T_x = \{1\}$. Let $x, y \in X$ be two distinct points of X. Applying Proposition 5.4(i), it follows that there is $a \in A$ such that $\varphi(a)(x) = 0_x$ and $\varphi(a)(y) = 1_y$. Hence, $a \in T_y$ and $a^- \in T_x$, since $\varphi(a^-)(x) = (\varphi(a)(x))^- = 0_x^- = 1_x$. We get that $0 = a \odot a^- \in T_x \vee T_y$, i.e. $T_x \vee T_y = A$. □ □

Proposition 5.6. *The following are equivalent:*
(i) φ is completely regular;
(ii) the family $(T_x)_{x\in X}$ is strongly coprime and the topology on X is generated by the family $V(a)_{a\in A}$.

Proof. (i)⇒(ii) By Proposition 5.4(ii), the topology on X is generated by the family $V(a)_{a\in A}$. Let us prove that the family $(T_x)_{x\in X}$ is strongly coprime. Let $x \in X$ and $a \in T_x$. Since $\varphi(A)$ is completely regular in (F, p, X) and $V(a)$ is an open neighborhood of x, there is $b \in A$ such that $\varphi(b)(x) = 0_x$ and $\varphi(b)(y) = 1_y$ for all $y \notin V(a)$. It follows that $b^- \in T_x$ and $b \in \bigcap\{T_y \mid y \in X \text{ and } a \notin T_y\}$, so $0 = b \odot b^- \in T_x \vee \bigcap\{T_y \mid y \in X \text{ and } a \notin T_y\}$.
(ii)⇒(i) Let $x \in X$ and U be an open neighborhood of x. Since the topology on X is generated by the family $V(a)_{a\in A}$, there is $a \in A$ such that $x \in V(a) \subseteq U$. We have that $T_x \vee \bigcap\{T_y \mid y \in X \text{ and } a \notin T_y\} = A$, so there are $b \in T_x$ and $c \in \bigcap\{T_y \mid y \in X \text{ and } a \notin T_y\}$ such that $b \odot c = 0$. Since $c \in T_y$ for all $y \in X$ such that $a \notin T_y$, we get that $\varphi(c)(y) = 1_y$ for all $y \notin V(a)$, hence $\varphi(c)(y) = 1_y$ for all $y \notin U$, since $V(a) \subseteq U$. From $b \odot c = 0$ we obtain that $b \leq c^-$, so $c^- \in T_x$, because $b \in T_x$ and T_x is a filter of A. We get that $\varphi(c^-)(x) = 1_x$, hence $\varphi(c)(x) = 0_x$. Thus, for any $x \in X$ and any open neighborhood U of x there is $c \in A$ such that $\varphi(c)(x) = 0_x$ and $\varphi(c)(y) = 1_y$ for all $y \notin U$. That is, $\varphi(A)$ is completely regular in (F, p, X). □ □

Theorem 5.7. *Let A be a nontrivial BL-algebra and $(T_x)_{x\in X}$ a filter space of A such that the subset $V(a) = \{x \in X \mid a \in T_x\}$ is open in X for all $a \in A$. The following are equivalent:*
(i) the filter space canonically determines a compact representation of A;
(ii) X is compact and the family $(T_x)_{x\in X}$ is coprime;
(iii) the family $(T_x)_{x\in X}$ is strongly coprime, the topology on X is generated by

the family $V(a)_{a\in A}$ and any maximal filter of A contains a filter of the filter space.

Proof. (i)⇒(ii) Obviously X is compact. Apply Lemma 5.5 to get that $(T_x)_{x\in X}$ are coprime.
(ii)⇒(i) Suppose that U is an open subset of X and let $x \in U$ and $C = U^c$. Then, for all $y \in C$ we have that $x \neq y$, so, by the fact that the family $(T_x)_{x\in X}$ is coprime, we obtain that $T_x \vee T_y = A$. Hence, for all $y \in C$, there are $a^y \in T_x$ and $b^y \in T_y$ such that $a^y \odot b^y = 0_y$. It follows that $(b^y)^- \in T_x$ for all $y \in C$. We also get that $y \in V(b^y)$ for all $y \in C$, so $C \subseteq \bigcup_{y\in C} V(b^y)$. Since C is a closed subset of the compact space X, C is also compact, hence there are $y_1, \cdots y_n \in C$ and $b_1 = b^{y_1}, \cdots b_n = b^{y_n} \in T_y$ such that $C \subseteq V(b_1) \cup \cdots \cup V(b_n)$. Let $b = b_1 \vee \cdots \vee b_n$. Then $b \in T_y$ for all $y \in C$, so $C \subseteq V(b)$, hence $V(b)^c \subseteq U$. We also have that $b^- = b_1^- \wedge \cdots \wedge b_n^- \in T_x$. Let us prove that $V(b^-) \subseteq U$. If $z \in V(b^-)$, then $b^- \in T_z$, so $b \notin T_z$, since T_z is proper. That is, $z \notin V(b)$, so $z \in U$. Thus, for any open subset U of X and any $x \in U$, we have got $b^x \in A$ such that $b^x \in T_y$ for all $y \notin U$, $(b^x)^- \in T_x$ and $V((b^x)^-) \subseteq U$. It follows that $U = \bigcup_{x\in U} V((b^x)^-)$, hence U is open in the representation topology. Thus, we have proved that the topology on X is generated by the family $V(a)_{a\in A}$. Let us now prove that the family $(T_x)_{x\in X}$ is strongly coprime. Let $x \in X$ and $a \in T_x$, i.e. $x \in V(a)$. Applying the above construction for $U = V(a)$, there is $b \in A$ such that $b^- \in T_x$ and $b \in T_y$ for all $y \notin V(a)$, so $b \in \bigcap\{T_y \mid y \in X \text{ and } a \notin T_y\}$. Hence, $0 = b \odot b^- \in T_x \vee \bigcap\{T_y \mid y \in X \text{ and } a \notin T_y\}$, that is, $T_x \vee \bigcap\{T_y \mid y \in X$ and $a \notin T_y\} = A$. Apply now Proposition 5.6 and the fact that X is compact to get (i).
(i)⇔(iii) Applying Theorem 5.2 and Proposition 5.6, it follows that the filters $(T_x)_{x\in X}$ canonically determine a completely regular representation of A, $\varphi : A \to \Gamma(X, F)$ iff the family $(T_x)_{x\in X}$ is strongly coprime and the topology on X is generated by the family $V(a)_{a\in A}$. Now, applying The compactness theorem we obtain that the representation φ is compact iff $\varphi(A)$ is compact in (F, p, X) iff every maximal filter of $\varphi(A)$ is fixed. Applying now Lemma 4.6(i) and the fact that $A \cong \varphi(A)$, we get that every maximal filter of $\varphi(A)$ is fixed iff any maximal filter of A contains a filter T_x for some $x \in X$. □ □

Applying Theorem 5.7, we prove the existence of a compact representation for any nontrivial BL-algebra A.

Proposition 5.8. *The family $(O(M))_{M\in Max(A)}$ canonically determines a compact representation of A.*

Proof. We have that $Max(A)$ is compact and Hausdorff and, applying Propositions 2.18(ii) and 2.19, it follows that the family $(O(M))_{M\in Max(A)}$ is coprime. It remains to prove that $V(a) = \{M \in Max(A) \mid a \in O(M)\}$ is open in $Max(A)$ for all $a \in A$. Let $M \in V(a)$. Then $a \in O(M)$, so there is $b \notin M$ such that $a \vee b = 1$. If $N \in d(b)$, then $b \notin N$ and $a \vee b = 1$, so $a \in O(N)$, that is $N \in V(a)$. Hence, $M \in d(b) \subseteq V(a)$, so $V(a)$ is open. □ □

Let $(F_A, p_A, Max(A))$ be the BL-sheaf space and $\varphi : A \to \Gamma(Max(A), F_A)$ the compact representation determined by the family $(O(M))_{M\in Max(A)}$. Then $(F_A)_M = A/O(M)$ for all $M \in Max(A)$, $p_A : F_A \to Max(A)$ is the canonical projection and $\varphi(a) = [a]$ for all $a \in A$, where $[a] \in \Gamma(Max(A), F_A)$ is defined by $[a](M) = a/O(M)$ for all $M \in Max(A)$.
Since, by Proposition 2.18(iii), $A/O(M)$ is a local BL-algebra, as a consequence of the above proposition and Proposition 5.1, we get the following result.

Corollary 5.9. [25]
Any non-trivial BL-algebra A is isomorphic with a subdirect product of local BL-algebras.

Proposition 5.10. $\varphi : A \cong \Gamma(Max(A), F_A)$.

Proof. We have to prove that $\varphi(A) = \Gamma(Max(A), F_A)$. Since $\varphi(A)$ is compact in $(F_A, p_A, Max(A))$, by Proposition 4.5, it is sufficient to show that every global section is locally in $\varphi(A)$. Let $\sigma \in \Gamma(Max(A), F_A)$. Then for all $M \in Max(A)$, $\sigma(M) \in A/O(M)$, so there is $a_M \in A$ such that $\sigma(M) = a_M/O(M) = [a_M](M) = \varphi(a_M)(M)$, so $M \in [\sigma = \varphi(a_M)]$. Thus, there is a family $(a_M)_{M\in Max(A)}$ of elements of A and a family $(U_M = [\sigma = \varphi(a_M)])_{M\in Max(A)}$ of open sets of $Max(A)$ such that $\sigma|_{U_M} = \varphi(a_M)|_{U_M}$ for all $M \in Max(A)$. That is, σ is locally in $\varphi(A)$. □ □

Example 5.11. *Let us consider the case when BL-algebra A is the interval $[0, 1]$ endowed with the structure induced by a continuous t-norm. Since A is a BL-chain, we get from Proposition 2.13 that A is local, so A has a unique maximal filter M. Hence, $Max(A) = \{M\}$ and it is easy to see that $O(M) = \{1\}$, so $A/O(M) = A/\{1\} \cong A$. Thus, the associated BL-sheaf space is $(F_A = [0, 1], p_A, Max(A) = \{M\})$, where $p_A : [0, 1] \to \{M\}$, $p_A(a) = M$. For all $a \in [0, 1]$, we have that $[a] : \{M\} \to [0, 1]$ is defined by $[a](M) = a$, so by the construction before Theorem 5.2, it follows that the topology on $F_A = [0, 1]$ is the discrete topology.*

6 The equivalence between BL-algebras and compact local BL-sheaf spaces

Let us denote by $CL - BL - ShSp$ the full subcategory of $BL - ShSp$ whose objects are compact local BL-sheaf spaces. By Proposition 3.5, there is a section functor $\mathcal{S} : BL - ShSp \to BL$. Then, by composing $\mathcal{S}$ with the inclusion functor, we get a functor from $CL - BL - ShSp$ to BL, denoted by $\mathcal{S}$, too.
In the sequel, we shall define a functor $\mathcal{T} : BL \to CL - BL - ShSp$ and we shall prove that the functors $\mathcal{S}, \mathcal{T}$ determine an equivalence between $CL-BL-ShSp$ and BL.
For any nontrivial BL-algebra A, let us define $\mathcal{T}(A) = (F_A, p_A, Max(A))$. By the previous section, $(F_A, p_A, Max(A))$ is a compact BL-sheaf space. For any

$M \in Max(A)$, we have that the stalk at M is $(F_A)_M = A/O(M)$. By Proposition 2.18(iii), $A/O(M)$ is a local BL-algebra, so $(F_A, p_A, Max(A))$ is a compact local BL-sheaf space.
Let A and B be nontrivial BL-algebras and $h : A \to B$ a BL-morphism. If M is a maximal filter of B, then $h^{-1}(M)$ is a maximal filter of A, by Proposition 2.10. Let us define $\overline{h} : Max(B) \to Max(A)$ by $(\overline{h})(M) = h^{-1}(M)$ for any maximal filter M of B.

Proposition 6.1. *Let $\overline{h} : Max(B) \to Max(A)$ be the function defined above. Then*
(i) $O(\overline{h}(M)) \subseteq h^{-1}(O(M)) \subseteq \overline{h}(M)$ for any maximal filter M of B;
(ii) $\overline{h}$ is continuous.

Proof. (i) Let $a \in O(\overline{h}(M))$, so there is $b \notin \overline{h}(M)$ such that $a \vee b = 1$. It follows that $h(a) \vee h(b) = 1$ and $h(b) \notin M$, since $b \notin h^{-1}(M)$. That is, $h(a) \in O(M)$, hence $a \in h^{-1}(O(M))$. By Proposition 2.19(i), $O(M) \subseteq M$, hence $h^{-1}(O(M)) \subseteq h^{-1}(M) = \overline{h}(M)$.
(ii) Let $M \in Max(B)$ and V be an open neighborhood of $\overline{h}(M)$. We shall prove that there is an open neighborhood U of M such that $\overline{h}(U) \subseteq V$, hence $\overline{h}$ is continuous at M. Since $\varphi : A \to (F_A, p_A, Max(A))$ is a completely regular representation of A, there is $a \in A$ such that $\varphi(a)(\overline{h}(M)) = 0/O(\overline{h}(M))$ and $\varphi(a)(N) = 1/O(N)$ for all $N \notin V$. Hence, $a/O(\overline{h}(M)) = 0/O(\overline{h}(M))$ and $a/O(N) = 1/O(N)$ for all $N \notin V$. Applying Proposition 2.7, we get that $a^- \in O(\overline{h}(M)$ and $a \in O(N)$ for all $N \notin V$. Since $O(\overline{h}(M)) \subseteq \overline{h}(M)$, we have that $a^- \in \overline{h}(M)$ and from the fact that $\overline{h}(M)$ is a maximal filter of A, it follows that $a \notin \overline{h}(M)$, hence $h(a) \notin M$. Thus, we have obtained that M is an element of the basic open set $U = d(h(a))$ of $Max(B)$. Let us prove that $\overline{h}(U) \subseteq V$. Suppose that there is $P \in U$ such that $\overline{h}(P) \notin V$. From $\overline{h}(P) \notin V$, it follows that $a \in O(\overline{h}(P))$, so, by(i), $a \in \overline{h}(P)$, that is $h(a) \in P$. This contradicts the fact that $P \in U$. Thus, $\overline{h}(U) \subseteq V$. □ □

Let $(\overline{h}^{-1}(F_A), q_A, Max(B))$ be the BL-sheaf space induced by the function $\overline{h} : Max(B) \to Max(A)$ and $(F_A, p_A, Max(A))$ and $i : F_A \to \overline{h}^{-1}(F_A)$ the canonical morphism over $\overline{h}$. Since $\overline{h} : Max(B) \to Max(A)$ is continuous, we get that $(\overline{h}, i) : (F_A, p_A, Max(A)) \to (\overline{h}^{-1}(F_A), q_A, Max(B))$ is a morphism of BL-sheaf spaces.

Proposition 6.2. *For any maximal filter $M \in Max(B)$, let us define $\psi_M : (\overline{h}^{-1}(F_A))_M \to (F_B)_M$, by $\psi_M(M, a/O(\overline{h}(M))) = h(a)/O(M)$ for any $a \in A$. Then $(1_{Max(B)}, \psi) : (\overline{h}^{-1}(F_A), q_A, Max(B)) \to (F_B, p_B, Max(B))$ is a morphism of BL-sheaf spaces.*

Proof. Firstly, let us prove that ψ_M is well-defined. Let $a, b \in A$ such that $a/O(\overline{h}(M)) = b/O(\overline{h}(M))$. It follows that $(a \to b) \odot (b \to a) \in O(\overline{h}(M))$, that is $(a \to b) \odot (b \to a) \in h^{-1}(O(M))$, so $(h(a) \to h(b)) \odot (h(b) \to h(a)) \in O(M)$. Thus, $h(a)/O(M) = h(b)/O(M)$. Now, we shall apply Proposition 3.4 to get that $(1_{Max(B)}, \psi)$ is a morphism of BL-sheaf spaces. By the definition of ψ, it follows immediately that $p_B \circ \psi = q_A$ and that $\psi_M : \{M\} \times A/O(\overline{h}(M)) \to B/O(M)$ is a BL-morphism for all $M \in Max(B)$. It remains to prove that ψ is continuous. By Proposition 3.1(vi), it is sufficient to prove that ψ is open. Since the family $\{d(a) \mid a \in A\}$ is a basis for $Max(A)$, by Proposition 3.1(iv), Proposition 5.10, it follows that a basis for the topology of F_A is the family $\{[c](d(a)) \mid a, c \in A\}$. We get that a basis for $\overline{h}^{-1}(F_A)$ is the family $\{(d(b) \times [c](d(a))) \cap \overline{h}^{-1}(A) \mid a, c \in A, b \in B\}$. A basic open set in $\overline{h}^{-1}(A)$ is $(d(b) \times [c](d(a))) \cap \overline{h}^{-1}(A) = \{(M, c/O(\overline{h}(M))) \mid M \in Max(B), b \notin M, a \notin \overline{h}(M)\} = \{(M, c/O(\overline{h}(M))) \mid M \in Max(B), b \notin M, h(a) \notin M\}$, where $a, c \in A$, $b \in B$. It follows that $\psi((d(b) \times [c](d(a))) \cap \overline{h}^{-1}(A)) = \{\psi(M, c/O(\overline{h}(M))) \mid M \in Max(B), b \notin M, h(a) \notin M\} = \{h(c)/O(M) \mid M \in Max(B), b \notin M, h(a) \notin M\} = [h(c)](d(b) \cap d(h(a)))$, which is open in F_B. Hence, ψ is open. □ □

Hence, for a BL-morphism $h : A \to B$, we have got the morphisms of BL-sheaf spaces $(\overline{h}, i) : (F_A, p_A, Max(A)) \to (\overline{h}^{-1}(F_A), q_A, Max(B))$ and $(1_{Max(B)}, \psi) : (\overline{h}^{-1}(F_A), q_A, Max(B)) \to (F_B, p_B, Max(B))$. We define $\mathcal{T}(h) = (1_{Max(B)}, \psi) \circ (\overline{h}, i) = (\overline{h}, \alpha_h) : \mathcal{T}(A) \to \mathcal{T}(B)$, where $\alpha_h = \psi \circ i$.

Thus, we completed the definition of the functor $\mathcal{T} : BL \to CL - BL - ShSp$.

Proposition 6.3. $\mathcal{S} \circ \mathcal{T} \cong 1_{BL}$.

Proof. For any BL-algebra A, we have that $(\mathcal{S} \circ \mathcal{T})(A) = \Gamma(Max(A), F_A)$ and for any BL-morphism $h : A \to B$, $(\mathcal{S} \circ \mathcal{T})(h) = \mathcal{S}(\overline{h}, \alpha_h) = \alpha_{h\#}$, where $\alpha_{h\#} : \Gamma(Max(A), F_A) \to \Gamma(Max(B), F_B)$ is the BL-morphism induced by α_h. By Proposition 5.10, we have an isomorphism $\varphi_A : A \cong \Gamma(Max(A), F_A)$ for any non-trivial BL-algebra A. Let us prove that $\varphi = (\varphi_A)_{A \in Ob(BL)} : 1_{BL} \cong \mathcal{S} \circ \mathcal{T}$ is a natural transformation. For any $a \in A$ and $M \in Max(B)$, we have that $((\alpha_{h\#} \circ \varphi_A)(a))(M) = ((\alpha_{h\#})[a])(M) = (\alpha_{hM})([a](\overline{h}(M))) = (\alpha_{hM})(a/O(\overline{h}(M))) = h(a)/O(M)$ and $((\varphi_B \circ h)(a))(M) = [h(a)](M) = h(a)/O(M)$. Thus, $(\mathcal{S} \circ \mathcal{T})(h) \circ \varphi_A = \varphi_B \circ h$. Hence, $\varphi : 1_{BL} \cong \mathcal{S} \circ \mathcal{T}$ is a natural isomorphism. □ □

Proposition 6.4. $\mathcal{T} \circ \mathcal{S} \cong 1_{CL-BL-ShSp}$.

Proof. Let (F, p, X) be a compact local BL-sheaf space and $A = \Gamma(X, F)$. Then $(\mathcal{T} \circ \mathcal{S})(F, p, X) = \mathcal{T}(A) = (F_A, p_A, Max(A))$. Let $\mathbf{n} : X \to Max(A)$ the function that associates with any $x \in X$ the unique maximal filter M of A such that $K_x \subseteq M$. By Proposition 4.17, $\mathbf{n}$ is a homeomorphism. Let $x \in X$ and $M = \mathbf{n}(x)$, so $\mathbf{m}(M) = x$. Then, by Proposition 4.12, we get that $O(M) = K_{\mathbf{m}(M)}$, that is $O(\mathbf{n}(x)) = K_x$. Applying now Proposition 4.2(iv), it follows that $A/O(\mathbf{n}(x)) \cong F_x$. If $\alpha_x : A/O(\mathbf{n}(x)) \to F_x$ is this isomorphism, then $\alpha_x(\sigma/O(\mathbf{n}(x))) = \sigma(x)$ for all $\sigma \in A$. Let us prove that $\alpha : F_A \to F$ is a morphism over $\mathbf{n}$. Let D be an open subset of $Max(A)$ and $t \in \Gamma(D, F_A)$. We have to prove that the function $\alpha^D_{\#}(t) : \mathbf{n}^{-1}(D) \to F$, defined by $\alpha^D_{\#}(t)(x) = \alpha_x(t(\mathbf{n}(x)))$ for any $x \in \mathbf{n}^{-1}(D)$, is continuous. Since $(F_A, p_A, Max(A))$ is compact and t is a section over an open subset D of $Max(A)$, we can apply Proposition 4.2(ii) to get an open covering $(D_i)_{i\in I}$ of D and a family $(t_i)_{i\in I}$ of sections from $\Gamma(Max(A), F_A)$ such that $t|_{D_i} = t_i|_{D_i}$ for all $i \in I$. Applying now Proposition 5.10, we obtain a family $(\sigma_i)_{i\in I}$ of sections from $\Gamma(X, F)$ such that $t_i = [\sigma_i]$ for all $i \in I$. Let $x \in \mathbf{n}^{-1}(D)$, that is $\mathbf{n}(x) \in D$. Then, there is $k \in I$ such that $\mathbf{n}(x) \in D_k$, so $t(\mathbf{n}(x)) = [\sigma_k](\mathbf{n}(x)) = \sigma_k/O(\mathbf{n}(x))$. It follows that $\alpha^D_{\#}(t)(x) = \alpha_x(\sigma_k/O(\mathbf{n}(x))) = \sigma_k(x)$. Let $V \subseteq F$ be an open neighborhood of $\sigma_k(x)$ and $U = \mathbf{n}^{-1}(D_k) \cap \sigma_k^{-1}(V)$. Then $U \subseteq \mathbf{n}^{-1}(D)$ is an open neighborhood of x. If $y \in U$, then $\mathbf{n}(y) \in D_k$ and $\sigma_k(y) \in V$. It follows that $t(\mathbf{n}(y)) = \sigma_k/O(\mathbf{n}(y))$, hence $\alpha^D_{\#}(t)(y) = \sigma_k(y) \in V$. Thus, $\alpha^D_{\#}(t)(U) \subseteq V$. Hence, we have proved that for any $x \in \mathbf{n}^{-1}(U)$ and for any open neighborhood V of $\alpha^D_{\#}(t)(x)$ there is an open neighborhood U of x such that $\alpha^D_{\#}(t)(U) \subseteq V$. That is, $\alpha^D_{\#}(t)$ is continuous.

Hence, $(\mathbf{n}, \alpha) : (F_A, p_A, Max(A)) \to (F, p, X)$ is an isomorphism of BL-sheaf spaces. Let $\lambda : \mathcal{T} \circ \mathcal{S} \to 1_{CL-BL-ShSp}$, where $\lambda_{(F,p,X)} = (\mathbf{n}, \alpha)$. It remains to prove that λ is a natural transformation. Let $(f, \beta) : (F, p, X) \to (G, q, Y)$ be a morphism of BL-sheaf spaces, $A = \Gamma(X, F)$ and $B = \Gamma(Y, G)$. Let $\lambda_{(F,p,X)} = (\mathbf{n_1}, \alpha_1)$, $\lambda_{(G,q,Y)} = (\mathbf{n_2}, \alpha_2)$. We have that $(\mathcal{T} \circ \mathcal{S})(f, \beta) = (\overline{\beta_{\#}}, \theta)$, where $\beta_{\#} : A \to B$, $\beta_{\#}(\sigma)(y) = \beta_y(\sigma(f(y)))$ for all $\sigma \in A$ and $y \in Y$, $\overline{\beta_{\#}} : Max(B) \to Max(A)$, $\overline{\beta_{\#}}(M) = (\beta_{\#})^{-1}(M)$, and
$\theta = (\theta_M)_{M\in Max(B)}$, $\theta_M : A/O(\overline{\beta_{\#}}(M)) \to B/O(M)$, $\theta_M(\sigma/(\overline{\beta_{\#}}(M))) = \beta_{\#}(\sigma)/O(M)$

for all $\sigma \in A$. We have to prove that $(f, \beta) \circ (\mathbf{n_1}, \alpha_1) = (\mathbf{n_2}, \alpha_2) \circ (\overline{\beta_\#}, \theta)$, that is $(\mathbf{n_1} \circ f, \beta \circ \alpha_1) = (\overline{\beta_\#} \circ \mathbf{n_2}, \alpha_2 \circ \theta)$. Let $y \in Y$. If $\sigma \in K_{f(y)}$, then $\sigma(f(y)) = 1_{f(y)}$, hence $\beta_\#(\sigma)(y) = \beta_y(\sigma(f(y))) = \beta_y(1_{f(y)}) = 1_y$, so $\beta_\#(\sigma) \in K_y \subseteq \mathbf{n_2}(y)$, that is $\sigma \in (\beta_\#)^{-1}(\mathbf{n_2}(y)) = \overline{\beta_\#}(\mathbf{n_2}(y))$. Thus, we have proved that $\overline{\beta_\#}(\mathbf{n_2}(y))$ is a maximal filter of A that contains $K_{f(y)}$. But $\mathbf{n_1}(f(y))$ is the unique maximal filter of A that contains $K_{f(y)}$. Hence, we must have $(\beta_\#)^{-1}(\mathbf{n_2}(y)) = \mathbf{n_1}(f(y))$. Let us prove now that $\beta \circ \alpha_1 = \alpha_2 \circ \theta$. Let $y \in Y$ and $\sigma \in A$. Then $(\beta \circ \alpha_1)_y(\sigma/O(\mathbf{n_1}(f(y)))) = \beta_y(\sigma(f(y))) = \beta_\#(\sigma)(y)$, and $(\alpha_2 \circ \theta)_y(\sigma/O(\overline{\beta_\#}(\mathbf{n_2}(y)))) = (\alpha_2)_y(\beta_\#(\sigma)/O(\mathbf{n_2}(y))) = \beta_\#(\sigma)(y)$. $\Box$ $\Box$

Thus, we have got

Theorem 6.5. *The functors $\mathcal{S} : CL - BL - ShSp \to BL$ and $\mathcal{T} : BL \to CL - BL - ShSp$ establish an equivalence between the category of nontrivial BL-algebras and the category of compact local BL-sheaf spaces.*

As a consequence, we get the corresponding result for MV-algebras.

Corollary 6.6. [9]
The functor from the category of compact local MV-sheaf spaces to the category of nontrivial MV-algebras, obtained by assigning to each compact local MV-sheaf space the MV-algebra of global sections, determines an equivalence between these categories.

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